

GAIN ESTIMATION METHODS FOR POLARIZED RADIO TELESCOPE ARRAYS

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In radio telescope arrays, the complex receiver gains and sensor noise powers are initially unknown and have to be calibrated. This can be done by observing a strong astronomical point source. Here we extend the calibration algorithms to the case of dual polarized arrays. The algorithms are based on factor analysis and eigenvalue decompositions. We show that a single observation does not provide a unique solution. With two observations of two point sources with known brightness matrices, the gains can be estimated up to two unknown phases, and with three observations the ambiguity reduces to a single unknown phase.

1. INTRODUCTION

In interferometric radio astronomy, the telescope array complex receiver gains and sensor noise powers are initially unknown and have to be calibrated. Gain calibration techniques for radio telescope systems have existed already for a long time [1][2]. However, with the advent of new generations of radio telescopes (the Low Frequency Array or LOFAR, and the Square Kilometer Array radio telescope or SKA), phased array beamforming issues receive renewed interest. Phased array beamforming and RFI suppression using spatial filtering techniques can benefit from accurate gain calibration of the telescope array.

For unpolarized telescope arrays, the standard calibration procedure is to point the telescopes to a strong astronomical source, and to estimate a covariance matrix $\hat{\mathbf{R}}$, containing all correlation products between the telescope output signals. Asymptotically, $\hat{\mathbf{R}}$ converges to its expected value \mathbf{R} which has the model $\mathbf{R} = \mathbf{g}\sigma_s^2\mathbf{g}^H + \mathbf{D}$. Here, σ_s^2 is the known source flux, \mathbf{g} is a vector containing the complex gains to be estimated, and \mathbf{D} is a diagonal matrix containing the unknown noise powers per antenna element (it is assumed that the noise power is uncorrelated from one antenna to another). This is essentially the model considered by [1][2]. Improved estimation algorithms using iterative and closed form least squares techniques have recently been derived [3]: by incorporating proper weighting, these methods are proved to be asymptotically statistically efficient [4].

For dual polarized telescope arrays, much less is known. In 1995, Hamaker et al. [5][6] developed a matrix formalism in which the polarization properties of the astronomical signals and their propagation through the ionosphere and the astronomical receiving instrument are efficiently incorporated. An iterative procedure similar to SelfCal is used to estimate the polarization gain coefficients, but it is not known to what solution it will converge.

In this paper we extend the scalar gain calibration methods of [3] [4] for the case of polarized arrays. Again, the idea is to point at a strong known polarized sky source, and to estimate the dual polarization complex gain factors as well as and the system noise powers from an observed covariance matrix, assuming that the astronomical source flux is known from tables for all polarization components. We will show that the solution is not unique: there is a remaining unknown 2×2 unitary factor. To reduce the non-uniqueness, a *second* observation of a known sky source with a different polarization is needed. This reduces the non-uniqueness to two unknown phases that scale the columns of the gain matrix. It may be possible to further reduce this to a single unknown phase by observing a *third* source, and this is the best that can be expected.

Notation H is the complex conjugate (Hermitian) transpose, t the matrix transpose, overbar $\bar{}$ the complex conjugate, and † the matrix pseudo inverse (Moore-Penrose inverse). $\mathcal{E}\{\cdot\}$ is the expectation operator. \mathbf{I} is the identity matrix.

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2. DATA MODEL

2.1. Coherency

In aperture synthesis radio astronomy, the output of the interferometers is the correlation of the field strengths at the different telescopes, also known as coherencies [7]. The electric field at the location of an antenna element can be described by two linear polarization components, stacked in a 2×1 vector: $\mathbf{e}_i = [e_{ix}, e_{iy}]^t$. The correlation between two different telescopes i and j is a 2×2 interferometer coherency matrix $E_{ij} = \mathcal{E}\{\mathbf{e}_i \mathbf{e}_j^H\}$. If there are p telescopes, each with two polarizations, then the $2p$ observed electric fields can similarly be stacked in one vector: $\mathbf{e} = (\mathbf{e}_1^t, \dots, \mathbf{e}_p^t)^t$. The $2p \times 2p$ Hermitian coherency matrix \mathbf{E} is defined by $\mathbf{E} = \mathcal{E}\{\mathbf{e} \mathbf{e}^H\}$ which can be written in terms of interferometer coherency matrices E_{ij} as

$$E_{i,j} = \mathcal{E}\{\mathbf{e}_i \mathbf{e}_j^H\} = \begin{bmatrix} \mathcal{E}\{e_{ix} \bar{e}_{jx}\} & \mathcal{E}\{e_{ix} \bar{e}_{jy}\} \\ \mathcal{E}\{e_{iy} \bar{e}_{jx}\} & \mathcal{E}\{e_{iy} \bar{e}_{jy}\} \end{bmatrix}, \quad \mathbf{E} = \mathcal{E}\{\mathbf{e} \mathbf{e}^H\} = \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1p} \\ E_{21} & E_{22} & \cdots & E_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ E_{p1} & E_{p2} & \cdots & E_{pp} \end{bmatrix}.$$

\mathbf{E} is dependent on frequency and time, but for our analysis we assume that we work in a narrow subband and estimate the coherencies at sufficiently short time scales.

2.2. Observed covariance matrix

Instead of the field strengths, each telescope measures a voltage vector \mathbf{v}_i . Their relation is given by $\mathbf{v}_i = J_i \mathbf{e}_i$, where J_i is a 2×2 matrix called the Jones matrix. It also incorporates the various ionospheric and atmospheric distortions, gain phase rotations and antenna feed polarization leakage. Hence, the J_i are unknown and have to be estimated.

The observed voltages of the dual polarization output signals of the telescopes i and j are cross-correlated into covariance matrices R_{ij} , for which $R_{ij} := \mathcal{E}\{\mathbf{v}_i \mathbf{v}_j^H\} = J_i E_{ij} J_j^H$. Stacking the telescope output voltages \mathbf{v}_i into a $2p$ -dimensional vector $\mathbf{v} = [\mathbf{v}_1^t, \dots, \mathbf{v}_p^t]^t$, and defining

$$\mathbf{J} = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{bmatrix}, \quad \mathbf{R} = \mathcal{E}\{\mathbf{v} \mathbf{v}^H\} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ R_{21} & R_{22} & \cdots & R_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p1} & R_{p2} & \cdots & R_{pp} \end{bmatrix},$$

it follows that the $2p \times 2p$ covariance matrix \mathbf{R} is given by $\mathbf{R} = \mathbf{J} \mathbf{E} \mathbf{J}^H$.

In practice the observations are corrupted by noise. The system noise signals of each of the two polarization channels, $\mathbf{n}_i = [n_{ix}, n_{iy}]^t$ are stacked into a vector: $\mathbf{n} = (\mathbf{n}_1^t, \dots, \mathbf{n}_p^t)^t$. The noise signals are uncorrelated between the telescopes, and up to a certain level also uncorrelated between the two polarizations of a telescope. In our analysis we assume that this is the case. Then the noise matrix $\mathbf{D} = \mathcal{E}\{\mathbf{n} \mathbf{n}^H\}$ is diagonal: $\mathbf{D} = \text{diag}(\sigma_{1x}^2, \sigma_{1y}^2, \dots, \sigma_{px}^2, \sigma_{py}^2)$. The system noise can be considered additive, so that the covariance matrix of the received data can be written as $\mathbf{R} = \mathbf{J} \mathbf{E} \mathbf{J}^H + \mathbf{D}$.

2.3. Point source model

Under certain conditions, the electric field can be modeled as the contributions of a finite number of point sources:

$$\mathbf{R} = \sum_{\ell} \mathbf{J}_{\ell} \mathbf{E}_{\ell} \mathbf{J}_{\ell}^H + \mathbf{D}$$

where ℓ is the source direction and \mathbf{E}_{ℓ} is the coherency due to a single source from direction ℓ . Suppose that the source has sky brightness \mathbf{B}_{ℓ} (a 2×2 matrix determined by the source flux polarization components or Stokes parameters). The relation of \mathbf{B}_{ℓ} to \mathbf{E}_{ℓ} can be written as $E_{ij,\ell} = w_{ij,\ell} \mathbf{B}_{\ell}$ where $w_{ij,\ell}$ is the phase shift due to the geometric delay in an interferometer pair i - j [7]. Let \mathbf{r}_i be the location in space of the i -th telescope, and write the baseline $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, then

$$w_{ij,\ell} = e^{-i \mathbf{r}_{ij}^t \boldsymbol{\lambda}^{-1}} = e^{-i(\mathbf{r}_i^t \boldsymbol{\lambda}^{-1} - \mathbf{r}_j^t \boldsymbol{\lambda}^{-1})} = e^{-i \mathbf{r}_i^t \boldsymbol{\lambda}^{-1}} e^{i \mathbf{r}_j^t \boldsymbol{\lambda}^{-1}} = w_{i,\ell} \bar{w}_{j,\ell}$$

where $\boldsymbol{\lambda}$ is the wavelength at the selected frequency, and $w_{i,\ell} = e^{-i \mathbf{r}_i^t \boldsymbol{\lambda}^{-1}}$ is the phase shift at a single telescope. Note that it is the same for the x and the y polarization of this telescope. Thus define

$$W_{i,\ell} = \begin{bmatrix} w_{i,\ell} & 0 \\ 0 & w_{i,\ell} \end{bmatrix}, \quad \mathbf{W}_{\ell} = [W_{1,\ell}^t, \dots, W_{p,\ell}^t]^t,$$

then $E_{ij,\ell} = W_{i,\ell} \mathbf{B}_{\ell} W_{j,\ell}^H$, and $\mathbf{E}_{\ell} = \mathbf{W}_{\ell} \mathbf{B}_{\ell} \mathbf{W}_{\ell}^H$. The overall observed point source model thus becomes

$$\mathbf{R} = \sum_{\ell} \mathbf{J}_{\ell} \mathbf{W}_{\ell} \mathbf{B}_{\ell} \mathbf{W}_{\ell}^H \mathbf{J}_{\ell}^H + \mathbf{D}. \quad (1)$$

3. GAIN CALIBRATION OBSERVATIONS

During a calibration observation, the telescopes are pointed at a single dominant point source in the sky, with known sky brightness. The sum in equation (1) is reduced to a single term. Because the geometry of the telescope array is known, the delay matrix \mathbf{W}_ℓ is known as well. We thus obtain the observation model

$$\mathbf{R} = \mathbf{G}\mathbf{B}\mathbf{G}^H + \mathbf{D}$$

where we defined the $2p \times 2$ gain matrix \mathbf{G} by $\mathbf{G} = \mathbf{J}\mathbf{W}$. Our objective is to estimate \mathbf{G} and \mathbf{D} , assuming that an estimate of \mathbf{R} and \mathbf{B} are available. Since \mathbf{W}_ℓ is known, \mathbf{J} is easily determined from \mathbf{G} . Alternatively, \mathbf{R} can be corrected in advance for \mathbf{W}_ℓ , after which we can assume without loss of generality that $\mathbf{W}_\ell = \mathbf{I}$ and that $\mathbf{G} = \mathbf{J}$ is direction-independent.

\mathbf{R} is estimated by an observation covariance matrix $\hat{\mathbf{R}}$, obtained by cross-correlation of N samples \mathbf{x}_n of the telescope output signal vector, $\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^H$. More in general, we will show that we need to consider at least two observations $\mathbf{R}_1, \mathbf{R}_2$ of two sources $\mathbf{B}_1, \mathbf{B}_2$ with different polarizations, and model $\mathbf{R}_1 = \mathbf{G}\mathbf{B}_1\mathbf{G}^H + \mathbf{D}$, $\mathbf{R}_2 = \mathbf{G}\mathbf{B}_2\mathbf{G}^H + \mathbf{D}$. A solution for \mathbf{G} and \mathbf{D} can be found by solving the following Least Squares minimization problem,

$$\{\mathbf{G}, \mathbf{D}\} = \underset{\mathbf{G}, \mathbf{D}}{\operatorname{argmin}} \|\hat{\mathbf{R}}_1 - (\mathbf{G}\mathbf{B}_1\mathbf{G}^H + \mathbf{D})\|_F^2 + \|\hat{\mathbf{R}}_2 - (\mathbf{G}\mathbf{B}_2\mathbf{G}^H + \mathbf{D})\|_F^2$$

4. ALGORITHMS FOR FACTOR ANALYSIS

As a first step, we consider a single covariance matrix \mathbf{R} , and algorithms to find the factors \mathbf{A} and \mathbf{D} from a model $\mathbf{R} = \mathbf{A}\mathbf{A}^H + \mathbf{D}$, where \mathbf{D} is diagonal and the number of columns of \mathbf{A} is equal to two. This is a rank-2 factor model as studied in statistics [9]. We present two computationally efficient techniques.

4.1. Alternating Least Squares

A straightforward technique to try to optimize a cost function over many parameters is to alternately minimize over a subset, keeping the remaining parameters fixed. In our case, assume at the k -th iteration that we have an estimate $\hat{\mathbf{D}}[k]$. The next step is to minimize the LS cost function with respect to the gain vector only:

$$\hat{\mathbf{A}}[k] = \underset{\mathbf{A}}{\operatorname{argmin}} \|\hat{\mathbf{R}} - \mathbf{A}\mathbf{A}^H - \hat{\mathbf{D}}[k]\|_F^2 \quad (2)$$

The minimum is found from the eigenvalue decomposition $\hat{\mathbf{R}} - \hat{\mathbf{D}}[k] = \mathbf{U}\mathbf{A}\mathbf{U}^H$, where the matrix $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_{2p}]$ contains the eigenvectors \mathbf{u}_i , and \mathbf{A} is a diagonal matrix containing the eigenvalues λ_i , sorting in descending order. The factor minimizing (2) is given by $\hat{\mathbf{A}}[k] = [\mathbf{u}_1 \lambda_1^{1/2} \quad \mathbf{u}_2 \lambda_2^{1/2}]$. The second step is minimizing with respect to the system noise matrix \mathbf{D} , keeping the gain vector fixed:

$$\hat{\mathbf{D}}[k+1] = \underset{\mathbf{D}}{\operatorname{argmin}} \|\hat{\mathbf{R}} - \hat{\mathbf{A}}[k]\hat{\mathbf{A}}[k]^H - \mathbf{D}\|_F^2 \quad (3)$$

where \mathbf{D} is constrained to be diagonal with nonnegative entries. The minimum is obtained by subtracting $\hat{\mathbf{A}}[k]\hat{\mathbf{A}}[k]^H$ from $\hat{\mathbf{R}}$ and discarding all off-diagonal elements: $\hat{\mathbf{D}}[k+1] = \operatorname{diag}(\hat{\mathbf{R}} - \hat{\mathbf{A}}[k]\hat{\mathbf{A}}[k]^H)$. The condition that the diagonal elements of $\hat{\mathbf{D}}[k+1]$ should be positive can be implemented by subsequently setting the negative entries at zero. The two minimizations steps (2) and (3) are repeated until the model error converges. Since each of the minimizing steps in the iteration loop reduces the model error, we obtain monotonic convergence to a local minimum. Although the iteration is very simple to implement, simulations indicate that convergence can be very slow, especially in the absence of a reasonable initial point.

4.2. Closed form approximation

We now set out to find a closed form estimate of \mathbf{A} , which recovers \mathbf{A} exactly when applied to \mathbf{R} (hence asymptotically for $\hat{\mathbf{R}}$). The crux of this method is the observation that the off-diagonal entries of $\mathbf{A}\mathbf{A}^H$ are equal to those of \mathbf{R} , and known, so that we only need to reconstruct the diagonal entries of $\mathbf{A}\mathbf{A}^H$. We further note that $\mathbf{A}\mathbf{A}^H$ is rank 2, so any submatrix of \mathbf{R} that does not contain elements from the main diagonal is also rank 2. This property can be used to estimate the ratio between any triplet of columns of \mathbf{R} away from the diagonal, and subsequently to estimate how the main diagonal of \mathbf{R} has to be changed so that the resulting \mathbf{R}' is rank 2, or $\mathbf{R}' = \mathbf{A}\mathbf{A}^H$. The gain factor \mathbf{A} can then be extracted by an eigenvalue decomposition.

To illustrate the idea, let (i, j, k) be a triplet of column indices, and let \mathbf{M} be a submatrix of \mathbf{R} consisting of columns (i, j, k) , and all rows with indices unequal to i, j, k . Then \mathbf{M} has 3 columns, and rank 2, so that there exists a vector $\mathbf{v} = [v_1, v_2, v_3]^T$ such that $\mathbf{M}\mathbf{v} = 0$. The vector can be found from an SVD of \mathbf{M} . It follows that $[r'_{ii}, r'_{ij}, r'_{ik}]\mathbf{v} = 0$, so that $r'_{ii} = -(r_{ij}v_2 + r_{ik}v_3)/v_1$. This estimate can be improved by considering all possible triplets containing i , and combining the ratios. After filling in all diagonal entries of \mathbf{R}' in this way, a rank-2 factorization of $\mathbf{R}' = \mathbf{A}\mathbf{A}^H$ provides an estimate for the factor \mathbf{A} . An estimate for \mathbf{D} is subsequently found from $\mathbf{R} - \mathbf{A}\mathbf{A}^H$.

5. ALGORITHMS FOR POLARIZATION GAIN ESTIMATION

5.1. One reference source

Consider a single source, $\mathbf{R} = \mathbf{G}\mathbf{B}\mathbf{G}^H + \mathbf{D}$, where \mathbf{R} has been estimated and \mathbf{B} is known from sky tables. Using factor analysis, we can find \mathbf{D} and a factor \mathbf{A} such that $\mathbf{R} = \mathbf{A}\mathbf{A}^H + \mathbf{D}$. However, \mathbf{A} is not unique: for any 2×2 unitary matrix \mathbf{Q} , we have $\mathbf{A}\mathbf{A}^H = (\mathbf{A}\mathbf{Q})(\mathbf{Q}^H\mathbf{A}^H)$. Hence, we can estimate \mathbf{A} only up to a unitary factor. It follows that $\mathbf{G} = \mathbf{A}\mathbf{Q}\mathbf{B}^{-1/2}$, where \mathbf{Q} is unknown. It is not possible to estimate \mathbf{G} in more detail using only a single reference source.

5.2. Two reference sources

With two reference sources, we have

$$\mathbf{R}_1 = \mathbf{G}\mathbf{B}_1\mathbf{G}^H + \mathbf{D} = (\mathbf{A}_1\mathbf{Q}_1)(\mathbf{Q}_1^H\mathbf{A}_1^H) + \mathbf{D}_1, \quad \mathbf{R}_2 = \mathbf{G}\mathbf{B}_2\mathbf{G}^H + \mathbf{D} = (\mathbf{A}_2\mathbf{Q}_2)(\mathbf{Q}_2^H\mathbf{A}_2^H) + \mathbf{D}_2.$$

\mathbf{R}_1 and \mathbf{R}_2 are observed, \mathbf{B}_1 and \mathbf{B}_2 are the known polarization matrices from the reference sources, and \mathbf{A}_1 and \mathbf{A}_2 are the estimated factors from factor analysis. Again, they are unique only up to unknown 2×2 unitary factors $\mathbf{Q}_1, \mathbf{Q}_2$.

A generalized eigenvalue decomposition of the pair $(\mathbf{B}_1, \mathbf{B}_2)$ provides the factorizations $\mathbf{B}_1 = \mathbf{M}\mathbf{\Lambda}_1\mathbf{M}^H$, $\mathbf{B}_2 = \mathbf{M}\mathbf{\Lambda}_2\mathbf{M}^H$, where \mathbf{M} is a square invertible matrix and $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2$ are positive diagonal matrices. It is assumed that the generalized eigenvalues are distinct. Inserting this in the model equations produces two estimates for \mathbf{G} :

$$\mathbf{G} = \mathbf{A}_1\mathbf{Q}_1\mathbf{\Lambda}_1^{-1/2}\mathbf{M}^{-1} = \mathbf{A}_2\mathbf{Q}_2\mathbf{\Lambda}_2^{-1/2}\mathbf{M}^{-1}$$

The latter equality relates \mathbf{Q}_1 to \mathbf{Q}_2 as $\mathbf{A}_1^\dagger\mathbf{A}_2 = \mathbf{Q}_1(\mathbf{\Lambda}_1^{-1/2}\mathbf{\Lambda}_2^{1/2})\mathbf{Q}_2^H$. This has the form of an SVD, and \mathbf{Q}_1 and \mathbf{Q}_2 can be computed as the left and right singular vectors of $\mathbf{A}_1^\dagger\mathbf{A}_2$. However, these are unique only up to an unknown diagonal phase matrix $\mathbf{\Phi}$. Hence we obtain

$$\mathbf{G} = (\mathbf{A}_1\mathbf{Q}_1)\mathbf{\Phi}(\mathbf{\Lambda}_1^{-1/2}\mathbf{M}^{-1}) =: \mathbf{M}_1\mathbf{\Phi}\mathbf{M}_2,$$

where only $\mathbf{\Phi}$ is unknown. Thus, we can estimate \mathbf{G} only up to two unknown phases. With some effort, this can be converted to a more convenient normalization: if we define a normalized \mathbf{G} to have positive real entries on its first row, then the normalized \mathbf{G} is unique. This is the best that can be expected using two reference sources.

If a third observation of a point source is available, then the ambiguity can be further reduced to a single common phase. Indeed,

$$\mathbf{R}_3 = (\mathbf{M}_1\mathbf{\Phi}\mathbf{M}_2)\mathbf{B}_3(\mathbf{M}_2^H\mathbf{\Phi}^H\mathbf{M}_1^H) + \mathbf{D} \quad \Leftrightarrow \quad \mathbf{M}_1^{-1}(\mathbf{R}_3 - \mathbf{D})\mathbf{M}_1^{-1} = \mathbf{\Phi}(\mathbf{M}_2\mathbf{B}_3\mathbf{M}_2^H)\mathbf{\Phi}^H$$

Only $\mathbf{\Phi} = \text{diag}[\phi_1, \phi_2]$ is unknown, and the $(1, 2)$ element of this expression determines the ratio $\phi_1\bar{\phi}_2$.

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