TOWARDS A THEORY OF ELECTRODYNAMICS ON FRACTALS: REPRESENTATION OF FRACTAL CURVES AND INTEGRAL ELECTROMAGNETIC EQUATIONS IN FRACTAL DOMAINS

M. Giona[1], A. Adrover[1], W. Arrighetti[2], P. Baccarelli[2], P. Burghignoli[2], P. De Cupis[2], A. Galli[2], and G. Gerosa[2]

[1] ***Centro Interuniversitario di Ricerca sui Sistemi Disordinati e sui Frattali***
Chemical Engineering Department, "La Sapienza" University of Rome
Via Eudossiana 18, 00184 Roma, Italy

[2] ***Electronic Engineering Department, "La Sapienza" University of Rome***
Via Eudossiana 18, 00184 Roma, Italy

ABSTRACT

This work presents a rigorous setting of integral equations on fractal wire antennas. The proposed approach is based on the extension of Iterated Function System theory to obtain a parametric representation of fractal curves with respect to a normalised curvilinear abscissa, and to define the tangential properties on almost everywhere nondifferentiable curves.

INTRODUCTION

Fractal shaped antennas and fractal miniaturized elements are becoming a promising new family of devices with a wealth of potentially useful applications [1]. Owing to the facility of manufacturing fractal devices (the metallisation process of fractal antennas up to a few iterations of the recursive process that generates the fractal structure is a relatively simple task), several different prototypical fractal antennas have been produced and analysed.

The experimental and computational researches on such structures still lack a solid theoretical counterpart, which should yield a comprehensive framework for the definition and the analysis of vector fields defined on fractal subsets. As a consequence of this, some fundamental issues of electromagnetic theory for fractal radiators are still unsolved (e.g., whether fractal dipolar radiators are subject to the same fundamental limits of Euclidean ones). Existing literature cannot be of help for solving these problems, since it is intrinsically limited to structures which are the initial iterations in the construction process of fractal structures (*prefractals*) [2,3].

The aim of this work is to propose a systematic theoretical analysis of direct and inverse electromagnetic problems on fractals. We limit here to fractal structures which are topologically equivalent to one-dimensional curves (*fractal wires*). Two main issues are analysed: *i)* parametric representation theory of fractal structures by means of Augmented Iterated Function Systems; *ii)* definition of vector fields on fractal domains and solution of inverse problems expressed as integral (scalar and vector) equations defined on fractal supports. It is important to stress out that the definition of vector fields on fractal structures involves delicate theoretical issues, related to the definition of tangency properties on fractals. This can be intuitively argued by considering tangential current distributions on wire antennas defined for structures, like fractal curves, for which tangent vectors cannot be defined almost anywhere in a classical way, and the definition of which requires nontrivial concepts of measure theory.

ITERATED FUNCTION SYSTEMS

The representation of fractal sets by means of Iterated Function Systems (IFS) is one of the most powerful tools in fractal theory.

An IFS with probability (IFSP) is a system $\{w_h(\boldsymbol{x}), p_h\}_{h=1}^{N}$ of N contractive transformations $w_h : \mathbb{R}^d \to \mathbb{R}^d$ ($\mathbb{R}^d$ is the Euclidean d-dimensional space) with the associated probability weights $p_h > 0$, $\sum_{h=1}^{N} p_h = 1$ [4].

Within the development of fractal antenna theory, IFS have been essentially applied to construct fractal curves (as a model of a fractal-shaped wire antenna), and to define their geometric fractal properties [2,3]. A fractal structure can be regarded as the fixed point of the operator $W : H(\mathbb{R}^d) \to H(\mathbb{R}^d)$ defined by

$$W(B) = \bigcup_{h=1}^{N} w_h(B), \; B \in H(\mathbb{R}^d) \tag{1}$$

where $H\left(\mathbb{R}^d\right)$ is the space of all the compact sets of $\mathbb{R}^d$; $H\left(\mathbb{R}^d\right)$ is a complete metric space once equipped with the Hausdorff metric d_H [4].

Owing to the contractive nature of the maps defining the IFS, the operator W turns out to be contractive in $\left(H\left(\mathbb{R}^d\right), d_H\right)$, so that its unique fixed point C

$$C = \bigcup_{h=1}^{N} w_h\left(C\right) \tag{2}$$

defines (Hutchinson condition) the fractal set generated by the IFS independently of the value of the probability weights p_h. IFS theory provides also a simple way to obtain a geometric representation (visualization) of the fractal set C, defined by (2), since C can be obtained as the limit set of the random iteration algorithm constructed starting from $\left\{w_h\left(\boldsymbol{x}\right), p_h\right\}_{h=1}^{N}$ as:

$$\boldsymbol{x}_{n+1} = w_h\left(\boldsymbol{x}_n\right) \tag{3}$$

where the next $\boldsymbol{x}_{n+1}$ is obtained as the image of the point $\boldsymbol{x}_n$ through one of the N maps w_h randomly selected with probability p_h.

Amongst all the possible IFS, self-similar IFS are particularly convenient in practical applications. For $d = 2$, a self-similar IFS takes the form $w_h\left(\boldsymbol{x}\right) = \boldsymbol{A}_h\boldsymbol{x} + \boldsymbol{b}_h$ ($h = 1,...,N$), where $\boldsymbol{A}_h$ are self-similar matrices obtained by the composition of similitudes $\boldsymbol{A} = s_h\boldsymbol{I}$ ($0 < s_h < 1$, $\boldsymbol{I}$ identity matrix), rotations of an angle θ_h, and mirror reflections around one of the two axes, while $\boldsymbol{b}_h$ are translation vectors. An example of fractal curve generated in this way is depicted in Fig. 1: it is obtained by means of an IFS with $N = 8$, where all the eight maps are characterised by the same scaling factor $s = 1/4$, so that its fractal (Hausdorff) dimension is $d_F = \log 8 / \log 4 = 3/2$.

AUGMENTED IFS AND WIRE-ANTENNA ANALYSIS

The IFS theory, briefly described in the previous section, is not sufficient to approach the general setting of integral equations for antenna analysis on fractal wire structures, and a further generalisation is needed. To see this, let us consider the classical thin-wire approximation. Let us consider a curve γ, representing the antenna (corresponding to a cylindrical structure of radius a, wrapped around γ), and parameterised with respect to a normalized curvilinear abscissa $s \in \left[0,1\right]$. The two basic equations, defining the direct and the inverse electromagnetic problem, are: *i)* the expression for the vector potential as a function of the current $I\left(s\right)$

$$\boldsymbol{A}\left(\boldsymbol{x}\right) = \mu\int_0^1 I\left(s'\right)\boldsymbol{t}\left(s'\right)g\left(\boldsymbol{x},\boldsymbol{x}\left(s'\right)\right)ds' \tag{4}$$

where $\boldsymbol{t}\left(s\right)$ is the unit vector tangent to γ and $g\left(\boldsymbol{x},\boldsymbol{x}\left(s'\right)\right)$ is free-space Green's function; *ii)* the Pocklington's equation defining $I\left(s\right)$ as a function of the impressed tangential electric field $\boldsymbol{E}_t^i\left(s\right)$

$$\int_0^1 I\left(s'\right)\left[\frac{\partial}{\partial s}\frac{\partial}{\partial s'} - k^2\boldsymbol{t}\left(s\right)\cdot\boldsymbol{t}\left(s'\right)\right]g\left(\boldsymbol{x}\left(s\right),\boldsymbol{x}\left(s'\right)\right)ds' = i\omega\varepsilon\boldsymbol{E}_t^i\left(\boldsymbol{x}\left(s\right)\right) \tag{5}$$

where $\partial/\partial s = \boldsymbol{t}\left(s\right)\cdot\nabla$.

In order to recast these two equations in a rigorous formal setting suitable for fractal structures, two fundamental ingredients are necessary: *i)* the parameterisation of a fractal curve with respect to a normalised curvilinear abscissa; *ii)* the definition of curvilinear integral of differential forms, such as $\int_\gamma f_1dx + f_2dy$, on fractal curves. Both these issues can be approached and solved by introducing the concept of Augmented IFS. This approach is a simpler and rigorous version of the results developed in [5].

Given a planar self-similar IFS generating a fractal curve $C \subseteq \mathbb{R}^2$, the *Augmented* IFS (AIFS) can be associated to it, expressed through the maps $\tilde{w}_h : \mathbb{R}^3 \to \mathbb{R}^3$ ($h = 1,..,N$):

$$\tilde{w}_h(z) = \begin{pmatrix} \sigma_h & 0 \\ 0 & \boldsymbol{A}_h \end{pmatrix} \begin{pmatrix} s \\ \boldsymbol{x} \end{pmatrix} + \begin{pmatrix} \omega_h \\ \boldsymbol{b}_h \end{pmatrix} \tag{6}$$

where $\boldsymbol{z} = (s, \boldsymbol{x})^T$ and

$$\sigma_h = \frac{s_h}{\sum_{k=1}^{N} s_k}, \qquad \omega_h = \frac{\sum_{k=1}^{h-1} s_k}{\sum_{k=1}^{N} s_k} \tag{7}$$

The AIFS is defined on $\mathbb{R}^3$, and its limit set C_a possesses the following properties: the extra coordinate $s \in [0,1]$ represents a natural parameterisation of the fractal curve; to any values of $s \in [0,1]$, there exists a unique value $\boldsymbol{x}(s) \in C$, so that it is possible to define a vector-valued function $\boldsymbol{x} = \boldsymbol{T}(s) = (T_1(s), T_2(s))^T$ mapping the interval $[0,1]$ into *C*. This geometric construction is depicted in Figures 1-4: Fig. 2 shows the limit set C_a, Fig. 1 shows the projection of C_a onto the plane $\boldsymbol{x}$, corresponding to the fractal curve *C*, while the graphs of the two functions $T_1(s)$ and $T_2(s)$ are the projections of C_a onto the planes s-x_1, s-x_2, respectively (see Figs. 3 and 4).
By applying the AIFS it is possible not only to achieve a parametric representation of the fractal curve *C* but also to introduce the concept of line integral of a continuous vector-valued function $\boldsymbol{f}(\boldsymbol{x})$ over *C*. The line integral of $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))^T$ over *C* is defined as the Stieltjes integral of ***f*** with respect to $(T_1(s), T_2(s))$, i.e.:

$$\int_C \boldsymbol{f}(\boldsymbol{x}) \cdot d\boldsymbol{x} = \int_0^1 \left[f_1(\boldsymbol{x}(s))\, dT_1(s) + f_2(\boldsymbol{x}(s))\, dT_2(s) \right] \tag{8}$$

The quantity at the right-hand side of (8) exists and attains the value

$$\int_C \boldsymbol{f}(\boldsymbol{x}) \cdot d\boldsymbol{x} = \lim_{M \to \infty} \frac{1}{M} \sum_{k=1}^{M} \left[f_1(\boldsymbol{x}_k)(x_{1,k+1} - x_{1,k}) + f_2(\boldsymbol{x}_k)(x_{2,k+1} - x_{2,k}) \right] \tag{9}$$

where $\{(\boldsymbol{x}_k, s_k)\}_{k=1}^{M}$ are an ensemble of points, ordered with respect to non-decreasing values of s_k, obtained through the Random Iteration Algorithm [4] associated to the AIFS with the value of the probability weights equal to σ_h.
In a similar way, the vector potential associated with a prescribed tangential current distribution on a fractal wire antenna can be expressed as:

$$\boldsymbol{A}(\boldsymbol{x}) = \mu \left[\int_0^1 I(s')\, g(\boldsymbol{x}, \boldsymbol{x}(s'))\, dT_1(s')\, \boldsymbol{e}_1 + \int_0^1 I(s')\, g(\boldsymbol{x}, \boldsymbol{x}(s'))\, dT_2(s')\, \boldsymbol{e}_2 \right] \tag{10}$$

where $\boldsymbol{e}_i$, $i = 1, 2$ are the unit tangent vectors of the coordinate axes of the plane containing *C*.

CONCLUDING REMARKS

The generalisation of Iterated Function Systems which generate fractal curves, through the concept of augmented IFS, provides a parametric representation of fractal curves suitable for setting in a rigorous way the electromagnetic integral equations for wire structures. The solution of the inverse problem on fractal wire antennas can be obtained starting from (5), e.g., by simply expanding the current $I(s)$, parameterised with respect to the normalised curvilinear abscissa *s*, by means of a suitable basis of functions (i.e., the classical moment method).

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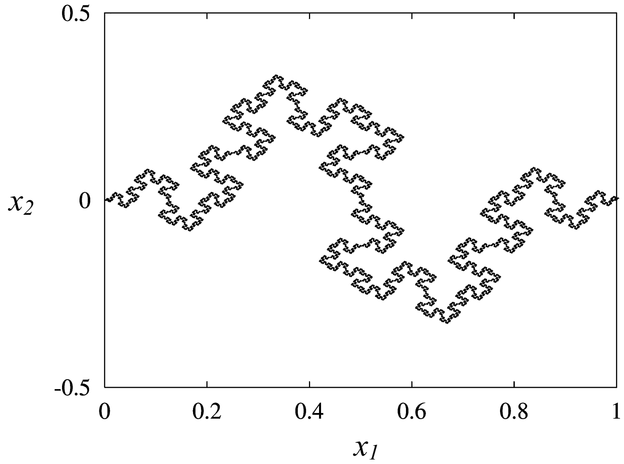

Fig. 1 - Attractor C of the IFS on the x_1-x_2 plane.

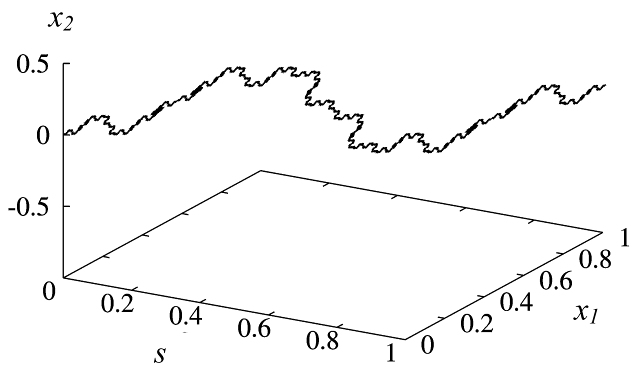

Fig. 2 - Attractor C_a of the AIFS, whose projection onto the $\boldsymbol{x}$ plane returns the attractor C of the IFS (see Fig. 1).

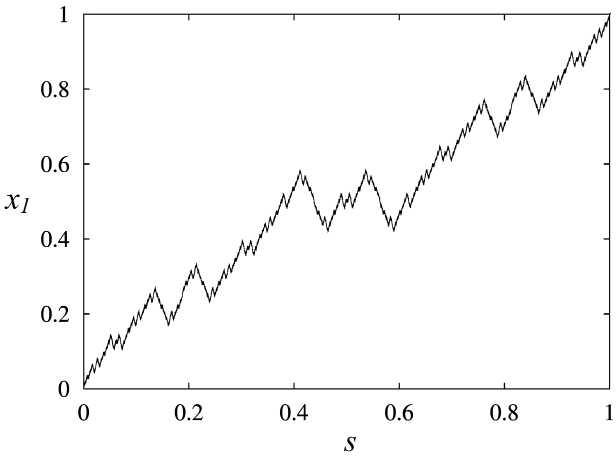

Fig. 3 - Graph of $T_1(s)$, projection of C_a onto the s-x_1 plane.

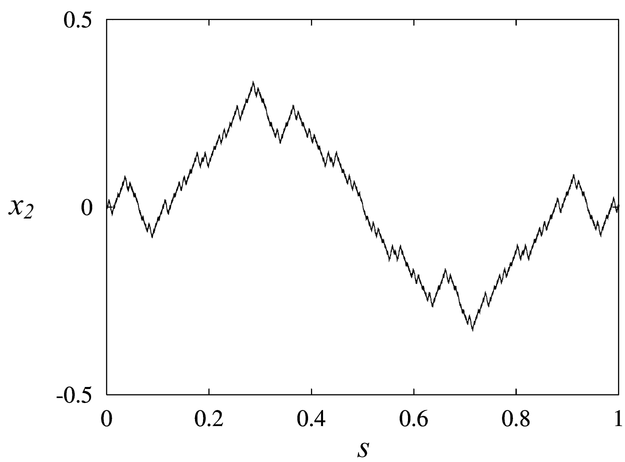

Fig. 4 - Graph of $T_2(s)$, projection of C_a onto the s-x_2 plane.