AN ANALYTIC PERFORMANCE ANALYSIS OF A CLASS OF PERFECTLY MATCHED LAYERS FOR TIME-DOMAIN ELECTROMAGNETCIC FIELD COMPUTATION

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ABSTRACT

A class of Perfectly Matched Layers (PML's), characterized by an excess time delay profile and an excess absorption profile, mimicking the reflectionless radiation into a homogeneous, isotropic, lossless embedding, is introduced and their performance in the presence of a truncated layer is analyzed for a test case (transient radiation from an electric-current loop) with a completely analytic time-domain solution. The thickness of the PML and the profile steering parameters admit adjustments to guarantee a desired accuracy in the computational target region of a finite-difference or finite-element code.

INTRODUCTION

Transient electromagnetic wave propagation and scattering problems are customarily analyzed in configurations of unbounded extent. The part of the configuration in which computational time-domain methods can be used to obtain the relevant field values is, however, necessarily of bounded support. This region, the *target region*, is taken to contain those parts of the configuration in which one is interested in the detailed behavior of the field quantities involved. The target region's *embedding* in $\mathcal{R}^3$ is, standardly, taken to have such simple physical properties that analytical representations can be constructed for the wave quantities in it. In principle, these representations can be employed to construct boundary relations on the outer boundary of the target region that mimick the (passive) radiation into the embedding without affecting, as far as possible, the computed field values in the target region itself. Through the construction of the embedding's Green's functions (wavefields excited by point sources of electric or magnetic volume current), the relevant boundary integral equations and Oseen's extinction theorem provide exact absorbing boundary conditions [1, Section 28.12]. Both of these relations do yield, on the target region's boundary, interrelations between the field quantities, but they do so in a non-local and non-instantaneous manner and, hence, ruin the computationally favored spatial band and explicit time structure of the algorithm (for example, the finite-difference time-domain one) for solving the relevant wave equations.

One way to preserve the computationally favored structure of the algorithm, is to construct *absorbing boundary conditions* (ABC's) that sufficiently accurately approximate the exact boundary relations by spatially local and timely instantaneous ones. Several of these are known in the literature. More recently, truncated *perfectly matched layers* (PML's) have been introduced to serve the purpose.

The performance of both ABC's and PML's is usually tested through purely numerical experiments. In such experiments the employed signatures (pulse shapes) of the sources may hide some of the features that are inherent to the approximation at hand. For a test case where the time-domain field quantities can be constructed with entirely analytical techniques the authors have carried out a performance analysis for a number of ABC's and PML's [2]; this paper also contains a number of references to the earlier literature. The present contribution focuses in more detail on a class of PML's that is characterized by an excess time delay profile and/or an excess absorption profile, where the excess profiles serve to mimick exactly the reflectionless or passive radiation into a homogeneous, isotropic, lossless embedding. The computationally required truncation of the PML gives, however, rise to a spuriously reflected wave that disturbs the computed field values in the target region. The magnitude of this disturbance can be controlled by the steering parameters in the excess time delay and excess absorption profiles. Owing to the fact that the expressions for the field values of the spuriously reflected wave are analytically known, the thickness of the truncated PML and its profile parameters can be selected such that a desired accuracy of the computed result in the target region can be guaranteed.

THE TEST CONFIGURATION

As test configuration we take an electric current carrying wire in the shape of a small planar loop that emits transient electromagnetic radiation into a homogeneous, isotropic medium with permittivity ϵ and permeability μ. The barycenter of the loop is located at $\{x = 0, y = 0, z = h > 0\}$, where $\{x, y, z\}$ are the coordinates with respect to an orthogonal, right-handed, Cartesian reference frame. The vectorial area of the loop is $\boldsymbol{A} = A\boldsymbol{i}_z$, where $\boldsymbol{i}_z$ is the unit vector in the z-direction (Fig. 1). Let $I = I(t)$ be the electric current in the loop, with t the time coordinate. Then, the electric field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{H}$ of the emitted electromagnetic field are given by ([1], Section 26.10)

$$\boldsymbol{E} = -\mu\,\partial_t \boldsymbol{\nabla} \times \boldsymbol{F}, \quad \boldsymbol{H} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - c^{-2}\partial_t^2 \boldsymbol{F}, \tag{1}$$

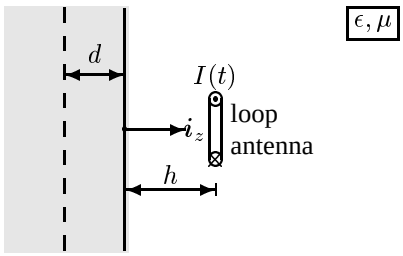

Figure 1: Test configuration: radiating loop antenna with perfectly matched layer in $\{z < 0\}$, truncated at $\{z = -d\}$.

where $\boldsymbol{F} = F\boldsymbol{i}_z$, with

$$F = \frac{AI(t - R_0/c)}{4\pi R_0} \quad \text{in which} \quad R_0 = [r^2 + (z-h)^2]^{1/2} \geq 0 \quad \text{and} \quad r^2 = x^2 + y^2. \tag{2}$$

Here, $AI(t)$ is the magnetic moment of the loop antenna and R_0 is the distance from the barycenter of the loop to the point of observation.

To have the radiation by this loop antenna serve as a test problem in the performance analysis of a PML, we provide the half-space $\{z < 0\}$ with the properties of such a layer. We shall carry out the construction of a rather general class of PML's through a time-domain coordinate stretching procedure. The starting point of this procedure is the partial differential equation satisfied by F, viz.

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 - c^{-2}\partial_t^2)F \;=\; -AI(t)\delta(x, y, z-h), \tag{3}$$

where $\delta(x, y, z-h)$ is the three-dimensional Dirac delta distribution operative at the point $\{x = 0, y = 0, z = h\}$.

THE TIME-DOMAIN COORDINATE STRETCHING PROCEDURE

The time-domain coordinate stretching procedure, carried out in the z-direction, consists of making, in the field equations, the following replacements:

$$\begin{aligned} &\partial_x \to \partial_x, \quad \partial_y \to \partial_y, \quad \partial_z \to \chi_z^{-1}(z,t) \overset{(t)}{*} \partial_z, \\ &\delta(x, y, z-z') \to \delta(x, y, z-z')\chi_z^{-1}(z,t) \overset{(t)}{*}, \end{aligned} \tag{4}$$

where $\chi_z^{-1}(z,t)$ is the inverse of the z-coordinate, *stretching function* $\chi_z(z,t)$ of the layer [3,4], i.e., $\chi_z^{-1}(z,t) \overset{(t)}{*} \chi_z(z,t) = \delta(t)$ for all $z \in \mathcal{R}$. To comply with the condition that the field in the half-space $\{z > 0\}$ (which is considered to be the target region of computation) should not be disturbed by the presence of the PML, we subject the stretching function to the condition $\chi_z(z,t) = \delta(t)$ for $z > 0$. Futhermore, the stretching function should satisfy the condition that the modified field equations in the configuration consisting of {target region} $\cup$ PML admit solutions that are uniquely determined by the excitation at hand, appropriate continuity conditions at interfaces and the property of causality in time. These aspects are most conveniently covered by an analysis in the time Laplace-transform domain.

THE COORDINATE STRETCHING PROCEDURE IN THE TIME LAPLACE-TRANSFORM DOMAIN

Assuming that the exciting electric current is switched on at the instant $t = 0$, the causal time Laplace transform $\hat{I} = \hat{I}(s)$ of $I = I(t)$ is given by

$$\hat{I}(s) \;=\; \int_{t=0}^{\infty} \exp(-st)I(t)\mathrm{d}t \quad \text{with } s \in \mathcal{R},\; s > 0. \tag{5}$$

We take the Laplace transform parameter s to be real and positive. (This implies that for the reconstruction of $I(t)$ from $\hat{I}(s)$ we have to rely on Lerch's uniqueness theorem [5], since the standard Bromwich inversion integral would require complex values of s.) Correspondingly, assuming zero-value initial conditions on the field quantities,

$$\hat{F}(x, y, z, s) \;=\; \int_{t=0}^{\infty} \exp(-st)F(x, y, z, t)\mathrm{d}t. \tag{6}$$

Since under the transformation $\hat{\partial}_t = s$, $\hat{F}$ satisfies the modified Helmholtz equation

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 - s^2/c^2)\hat{F} \;=\; -A\hat{I}(s)\delta(x, y, z-h), \tag{7}$$

while the application of the shift rule to (2) leads to

$$\hat{F} \;=\; A\hat{I}(s)\frac{\exp(-sR_0/c)}{4\pi R_0}. \tag{8}$$

On account of (4), the z-coordinate stretched differential equation satisfied by the coordinate stretched counterpart $\hat{F}^\pi$ of $\hat{F}$ follows as

$$\left\{\partial_x^2+\partial_y^2+\frac{1}{\hat{\chi}_z(z,s)}\partial_z\left[\frac{1}{\hat{\chi}_z(z,s)}\partial_z\right]-\frac{s^2}{c^2}\right\}\hat{F}^\pi \;=\; -A\hat{I}(s)\frac{1}{\hat{\chi}_z(z,s)}\delta(x,y,z-h). \tag{9}$$

To comply with the condition that the field equations in the stretched-coordinate domain admit causal solutions, we now assume that $\hat{\chi}_z(z,s)$ is, for all z, a function of s that is regular in some right half of the complex s-plane, bounded as $|s|\to\infty$ in this half-plane, and real and positive at the real s-axis in this half-plane. (Note that these conditions are the same as those applying to the transfer functions of linear, passive, time-invariant systems.) The solution of (9) that is bounded as $(x^2+y^2+z^2)^{1/2}\to\infty$ (condition of causality) is then given by

$$\hat{F}^\pi = A\hat{I}(s)\frac{\exp(-s\hat{R}_0^\pi/c)}{4\pi\hat{R}_0^\pi} \;\text{ with }\; \hat{R}_0^\pi = [r^2+\hat{Z}(z,h,s)^2]^{1/2} \;\text{ and }\; \hat{Z}(z,z',s)=\int_{\zeta=z'}^{z}\hat{\chi}_z(\zeta,s)\mathrm{d}\zeta. \tag{10}$$

Here, $\hat{Z}$ is the s-domain PML stretched z-coordinate from z' to z. Owing to the condition that $\hat{\chi}(z,s)=1$ for $z>0$, $\hat{F}^\pi$ reduces to $\hat{F}$ in the half-space $\{z>0\}$, which implies that the perfectly matched layer in the half-space $\{z<0\}$ as such does not influence the original wavefield in the target region $\{z>0\}$.

EXCESS TIME-DELAY AND EXCESS ABSORPTION PML PROFILES

The class of PML's whose performance we are going to analyze further has profiles of the type

$$\hat{\chi}_z(z,s) \;=\; 1+N(z)+s^{-1}\sigma(z), \tag{11}$$

which implies

$$\chi_z(z,t) \;=\; [1+N(z)]\delta(t)+\sigma(z)H(t) \text{ and } \chi_z^{-1}(z,t)=\frac{1}{1+N(z)}\delta(t)-\frac{\sigma(z)}{[1+N(z)]^2}\exp\left[-\frac{\sigma(z)}{1+N(z)}t\right]H(t), \tag{12}$$

where $H(t)$ denotes the Heaviside unit step function. In these expressions the *excess time-delay profile* $N=N(z)$ and the *excess absorption profile* $\sigma=\sigma(z)$ are non-negative functions of z for $z<0$ and vanish for $z>0$. (Note that such $\hat{\chi}_z(z,s)$, and correspondingly $\chi(z,t)$, satisfy the conditions for causality.) Substitution of (11) in (10) leads to

$$\hat{F}^\pi \;=\; sA\hat{I}(s)\hat{G}^\pi \quad \text{in which} \quad \hat{G}^\pi = \frac{1}{4\pi R_d}\,\frac{\exp\{-T_d[(s+\Gamma)^2+\Omega^2]^{1/2}\}}{[(s+\Gamma)^2+\Omega^2]^{1/2}} \quad \text{for } R_d\neq 0, \tag{13}$$

with

$$\begin{aligned} &Z_d=\int_{z'}^{z}[1+N_z(\zeta)]\mathrm{d}\zeta, \qquad Z_a=\int_{z'}^{z}\sigma_z(\zeta)\mathrm{d}\zeta, \qquad R_d=\left(r^2+Z_d^2\right)^{1/2}\geq 0, \qquad T_d=R_d/c,\\ &R_a=\left(r^2+Z_a^2\right)^{1/2}\geq 0, \qquad \Gamma=Z_dZ_a/R_d^2, \qquad \Omega=\left(R_a^2/R_d^2-\Gamma^2\right)^{1/2}\geq 0. \end{aligned} \tag{14}$$

The time-domain equivalent of (13) is [6]

$$F^\pi \;=\; \partial_t\left[AI(t)\overset{(t)}{*}G^\pi\right] \;\text{ in which }\; G^\pi=\frac{\exp(-\Gamma t)}{4\pi R_d}J_0[\Omega(t^2-T_d^2)^{1/2}]H(t-T_d) \;\text{ for } R_d>0, \tag{15}$$

where J_0 is the Bessel function of the first kind and order zero. From this it is clear that T_d is the travel time of the coordinate-stretched wave function from the source point to the point of observation, Γ is the attenuation that the wave undergoes during its passage, while Ω is the angular frequency of oscillation induced by the coordinate stretching procedure.

THE PML TRUNCATION GENERATED SPURIOUSLY REFLECTED WAVE

To investigate the influence of a (computationally required) truncation of the PML, we determine the vector potential $F^{r;\pi}$

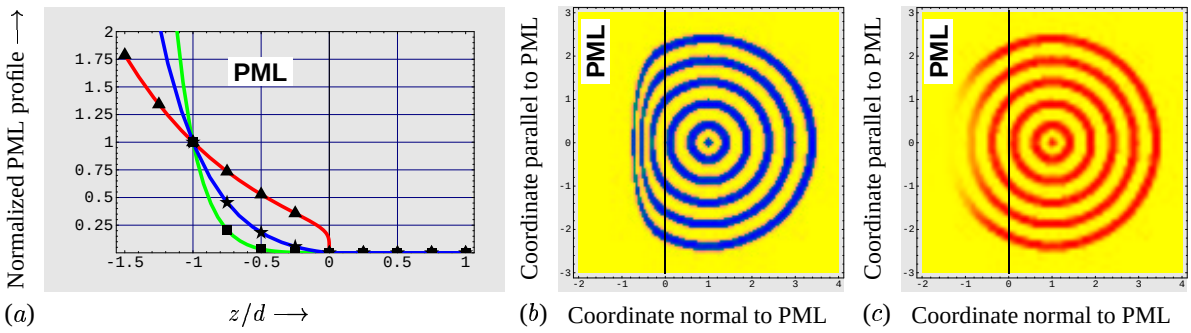

Figure 2: (a) Normalized PML excess time-delay and absorption profiles (▲ ▬: $A = 1.0$, $\nu = 0.5$, $\beta = 1.0$; ■ ▬: $A = 1.0$, $\nu = 2.0$, $\beta = 4.0$; ⋆ ▬: $A = 1.0$, $\nu = 1.0$, $\beta = 2.0$); (b) Wavefronts at successive time intervals in configuration with excess time delay PML ($A_N = 4.0$, $\nu_N = 1.0$, $\beta_N = 1.0$; $A_\sigma = 0$); (c) Wavefronts at successive time intervals in configuration with excess absorption PML ($A_\sigma = 4.0$, $\nu_\sigma = 1.0$, $\beta_\sigma = 1.0$; $A_N = 0$).

of the spuriously reflected wave generated by a planar boundary at $z = -d$, with $d > 0$, on which the tangential electric field strength vanishes, i.e., $F^\pi + F^{r;\pi} = 0$ as $z \downarrow -d$. Application of the method of images leads to

$$\hat{F}^{r;\pi} \;=\; -\frac{A\hat{I}(s)}{4\pi\hat{R}_2^\pi}\exp[-(s/c)\hat{R}_2^\pi] \quad \text{for } z > -d, \tag{16}$$

in which

$$\hat{R}_2^\pi = [r^2 + \hat{Z}_2^2]^{1/2} \qquad \text{with} \qquad \hat{Z}_2(z,d,h,s) = \int_{\zeta=-(2d+h)}^{z} \hat{\chi}_z'(\zeta,s)\mathrm{d}\zeta. \tag{17}$$

Here $\hat{\chi}_z'$ is the s-domain PML profile symmetrized about the plane $\{z = -d\}$ of truncation and $\hat{Z}_2$ is the s-domain PML stretched coordinate normal to the layer from the image of the source in the truncation plane to the point of observation in the half-space $\{z \geq 0\}$. Transformation of (16) back to the time domain yields the expression for $F^{r;\pi} = F^{r;\pi}(x,y,z,t)$. PML profiles characterized by an excess time-delay constituent and an excess absorption constituent, lead to an expression for $F^{r;\pi}$ of the type (15), in which the stretched coordinate quantities now apply to the path traversed from the image of the source to the point of observation via the symmetrized profile with the plane $\{z = -d\}$ as the plane of symmetry.

SOME FULLY ANALYTICALLY AMENABLE EXAMPLES

Illustrative examples have been worked out for profiles of the type

$$\{N(z), \sigma(z)\} \;=\; \{A_{N,\sigma}(-z/d)^{\nu_{N,\sigma}}\exp[-\beta_{N,\sigma}(z+d)], 0\} \;\text{ for }\; \{-d < z < 0, z > 0\}, \tag{18}$$

where $A_{N,\sigma}$, $\nu_{N,\sigma}$ and $\beta_{N,\sigma}$ are real-valued, non-negative parameters. At the reference plane $\{z = -d\}$, these profiles have the value $A_{N,\sigma}$, while they are continuous across the plane $\{z = 0\}$ where the PML starts. This continuity requirement seems to be preferred in computational implementations, although our analysis does not require it. The value of $\nu_{N,\sigma}$ determines the behavior near $\{z = 0\}$, the value of $\beta_{N,\sigma}$ determines the behavior as $z \to -\infty$. Figure 2a illustrates this. Figure 2b shows how the propagation of the disturbance into the PML with excess time delay is slowed down. Figure 2c indicates how the disturbance is attenuated with increasing depth in the PML. All the different steering parameters in (18) are at one's disposal to construct PML's with a guaranteed time delay and/or attenuation in the computational target region.

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