

# APPLICATION OF COMBINED FIELD INTEGRAL EQUATION FOR ELECTROMAGNETIC SCATTERING PROBLEMS

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## Abstract

In this paper a solution of combined field integral equation (CFIE) for electromagnetic scattering by an arbitrary shaped dielectric object is studied. We derive a novel CFIE formulation in which the electric and magnetic field integral equations (EFIE and MFIE) are combined in the usual manner outside the object. Inside the object CFIE is composed by changing the roles of EFIE and MFIE. In addition, we consider numerical evaluation of the singular impedance matrix elements with the singularity extraction technique.

## 1 INTRODUCTION

There are several alternative ways to formulate boundary integral equations for electromagnetic scattering by homogeneous dielectric bodies. The most widely applied formulations are the PMCHW formulation and the combined field formulation (CFIE) [1], [2]. In [2], the authors observed that if the RWG functions [3] are taken as both basis and test functions in the method of moments solution of the CFIE, the traditional form of CFIE leads to a very unstable solution. The reason is that only the electric surface current is well tested. As a remedy they suggest to test by both RWG and  $\mathbf{n} \times \text{RWG}$  functions, where  $\mathbf{n}$  is the outer unit normal of the object. The most convenient formulation was obtained when the electric field part of CFIE is tested by  $\text{RWG} + \mathbf{n} \times \text{RWG}$  functions and the magnetic field part is tested by RWG functions. This formulation was named a TENENH formulation.

An application of the method of moments with Galerkin's method to solve electromagnetic integral equations requires calculation of double integrals with singular kernels. Singular terms can be considered either by numerical methods (e.g. Duffy's method) or by singularity extraction technique [4]. In many cases numerical accuracy of the singular integrals is crucial for having an efficient algorithm, especially in the near field computing.

The aim of this paper is twofold. Firstly we present a new type of CFIE formulation. In this formulation we use traditional CFIE outside the object but inside the object the electric and magnetic field integral equations (EFIE and MFIE) are combined by changing their roles. This formulation gives TENH formulation outside the object and NETH formulation inside the object. This formulation has several nice properties: It is formally simpler than the TENENH formulation, it is free of interior resonances and it gives a stable solution. Secondly we present accurate and robust methods to evaluate singular integrals appearing in the CFIE formulations, when the integral equations are tested by both RWG and  $\mathbf{n} \times \text{RWG}$  functions and the basis functions are RWG functions. Our method is based on the singularity extracting technique. First we extract enough terms from the singular kernel so that the remaining function is at least once continuously differentiable and allows numerical integration. In evaluating the gradient of the Green's function the accuracy is easily lost although the singularity has been extracted. The reason is that in computing the impedance matrix elements we have to consider double integrals and the remaining outer integral may still have a logarithmical singularity. Therefore, in the terms including the gradient we modify the integrand and change the order of integration. By these modifications we can integrate all singular functions in closed form and the remaining terms are regular enough for numerical integration. Thus, Duffy's transformation or any other special integration quadratures are not required. Developed formulas and formulations are verified by considering numerical examples.

## 2 FORMULATION

Consider the problem of electromagnetic scattering by a homogeneous dielectric body  $D$  in  $\mathbb{R}^3$ . Let  $S$  denote the surface of  $D$  and let  $\mathbf{n}$  denote the outer unit normal of  $D$ . The traditional form of CFIE reads [1]

$$a \text{EFIE} + b \mathbf{n} \times \text{MFIE}, \quad (1)$$

where  $a = \alpha$ ,  $b = (1 - \alpha)\eta$ ,  $0 < \alpha < 1$ ,  $\eta = \sqrt{\mu_0/\varepsilon_0}$  and EFIE and MFIE denote the electric and magnetic field integral equations. An other possible CFIE formulation is to combine EFIE and MFIE as

$$a \mathbf{n} \times \text{EFIE} + b \text{MFIE}. \quad (2)$$

Using RWG functions [3] as both basis and test functions to discretize (1) and (2), gives the following equations

$$\begin{aligned} a \int_S \mathbf{f}_m \cdot \text{EFIE} \, dS - b \int_S \mathbf{n} \times \mathbf{f}_m \cdot \text{MFIE} \, dS, \\ -a \int_S (\mathbf{n} \times \mathbf{f}_m) \cdot \text{EFIE} \, dS + b \int_S \mathbf{f}_m \cdot \text{MFIE} \, dS, \end{aligned}$$

for all  $m = 1, \dots, N$ , where  $\mathbf{f}_m$  denotes an RWG function. Equation (1) gives a TENH formulation and (2) gives a NETH formulation [2]. As pointed out in [2] none of these two equations lead to a stable solution. As a remedy, in [2] the authors suggest to test EFIE by  $\mathbf{f}_m + \mathbf{n} \times \mathbf{f}_m$ . The resulting formulations are called TENENH and TENETH, respectively. However, the testing procedure can be simplified if EFIE and MFIE are combined as follows

$$\begin{aligned} a \text{EFIE-O} + b \mathbf{n} \times \text{MFIE-O} \\ a \mathbf{n} \times \text{EFIE-I} + b \text{MFIE-I}. \end{aligned} \quad (3)$$

Here O stands for outside  $D$  and I stands for inside  $D$ . This formulation gives TENH formulation outside  $D$  and NETH formulation inside  $D$ , so called TENH/NETH formulation. In TENH/NETH formulation the electric current is well tested by the TENH equation outside  $D$  and the magnetic current is well tested by the NETH equation inside  $D$ . Thus, both currents will be well tested. Furthermore, because both TENH and NETH formulations are free of interior resonances [2] also their combination shares the same property.

## 3 NUMERICAL IMPLEMENTATION

Let  $\mathbf{f}_n$  denote an RWG function with the support  $S_n$  and let  $\mathbf{g}_m$  denote an RWG function or a  $\mathbf{n} \times \text{RWG}$  function with the support  $S_m$ . An application of the method of moments to solve CFIEs (1) and (2) by the RWG functions requires calculation of the following double integrals [2]

$$\begin{aligned} I_1 &:= \int_{S_m} \mathbf{g}_m(\mathbf{r}) \cdot \nabla \int_{S_n} G(\mathbf{r}, \mathbf{r}') \nabla'_s \cdot \mathbf{f}_n(\mathbf{r}') \, dS' \, dS, \\ I_2 &:= \int_{S_m} \mathbf{g}_m(\mathbf{r}) \cdot \int_{S_n} G(\mathbf{r}, \mathbf{r}') \mathbf{f}_n(\mathbf{r}') \, dS' \, dS, \\ I_3 &:= \int_{S_m} \mathbf{g}_m(\mathbf{r}) \cdot \int_{S_n} \nabla G(\mathbf{r}, \mathbf{r}') \times \mathbf{f}_n(\mathbf{r}') \, dS' \, dS, \\ I_4 &:= \int_{S_m} \mathbf{g}_m(\mathbf{r}) \cdot (\mathbf{n}(\mathbf{r}) \times \mathbf{f}_n(\mathbf{r})) \, dS, \end{aligned}$$

where  $G$  is the free space Green's function.

If the supports of  $\mathbf{g}_m$  and  $\mathbf{f}_n$  are far away from each other, the above integrals are regular and they can be calculated numerically. Let us consider the case when  $S_m$  and  $S_n$  are close to each other or they have common

points and let  $R = |\mathbf{r} - \mathbf{r}'|$ . Because the RWG functions are composed on a triangle pair it suffices to consider calculation of the above integrals over single triangles. We begin by modifying integral  $I_1$  as

$$I_1 = \begin{cases} - \int_{T_m} \nabla_s \cdot \mathbf{f}_m(\mathbf{r}) \int_{T_n} G(\mathbf{r}, \mathbf{r}') \nabla'_s \cdot \mathbf{f}_n(\mathbf{r}') dS' dS, & \text{if } \mathbf{g}_m = \mathbf{f}_m, \\ \int_{\partial T_m} \mathbf{m}(\mathbf{r}) \cdot (\mathbf{n}(\mathbf{r}) \times \mathbf{f}_m(\mathbf{r})) \int_{T_n} G(\mathbf{r}, \mathbf{r}') \nabla'_s \cdot \mathbf{f}_n(\mathbf{r}') dS' dl, & \text{if } \mathbf{g}_m = \mathbf{n} \times \mathbf{f}_m, \end{cases}$$

where  $\mathbf{m}$  is the unit vector of the boundary  $\partial T_m$  of  $T_m$  pointing into the exterior of  $T_m$ . We may conclude the following:

- Integrals  $I_1$  and  $I_2$  have a singularity of order  $1/R$ .
- After evaluating the inner integrals of  $I_1$  and  $I_2$  with respect to  $\mathbf{r}'$ , the outer integral with respect to  $\mathbf{r}$  is regular.
- Integral  $I_3$  vanishes if  $S_m$  and  $S_n$  are on the same plane.
- If  $S_m$  and  $S_n$  are not in the same plane, integrand of  $I_3$  has a singularity of order  $1/R^2$ .
- If  $S_m$  and  $S_n$  are not in the same plane, after evaluated the inner integral of  $I_3$ , the outer integral may still have a logarithmic singularity.
- Integral  $I_4$  is regular.

Thereafter integrals  $I_1$  and  $I_2$  are calculated by extracting two terms from the Green's function

$$G(\mathbf{r}, \mathbf{r}') = \left( G(\mathbf{r}, \mathbf{r}') - \frac{1}{4\pi R} + \frac{k^2 R}{8\pi} \right) + \frac{1}{4\pi R} - \frac{k^2 R}{8\pi}.$$

Here the term in the brackets has two continuous derivatives and a standard integration routine on triangles such as the Gaussian quadrature gives accurate results. The last two terms on the right hand side (RHS) can be integrated in closed form [4], [5]. Consider next term  $I_3$  with an RWG testing function. Using the same idea as above, we write [6]

$$\nabla G(\mathbf{r}, \mathbf{r}') = \nabla \left( G(\mathbf{r}, \mathbf{r}') - \frac{1}{4\pi R} + \frac{k^2 R}{8\pi} \right) + \frac{1}{4\pi} \nabla \frac{1}{R} - \frac{k^2}{8\pi} \nabla R. \quad (4)$$

The first term on the RHS has a continuous derivative and therefore, it can be integrated numerically. The extracted terms, i.e. the second and third terms on the RHS of (4), can be integrated analytically over  $T_n$  using the formulas presented in [7] and [5]. Thereafter, the last term can be integrated numerically over  $T_m$ , because the integrand is regular. The problem is to integrate the second term over  $T_m$ . Let us consider this term more carefully.

Since an RWG function is given as  $\pm L/(2A)(\mathbf{r} - p)$  [3], where  $L$  and  $A$  are the length of an edge and the area of a triangle, it suffices to consider the following singular double integral

$$C_{mn} \int_{T_m} (\mathbf{r} - p) \cdot \int_{T_n} \nabla G_0(\mathbf{r}, \mathbf{r}') \times (\mathbf{r}' - q) dS' dS, \quad (5)$$

where  $G_0 = 1/(4\pi R)$ ,  $C_{mn} = \pm L_m L_n / (4A_m A_n)$  and  $p$  and  $q$  are the free vertices of the test and basis triangles. We consider this integral in two parts by separating the normal and surface derivatives. Before applying analytical formulas, we apply the Gauss divergence theorem to the surface gradient term and translate the integral over  $T_n$  into a line integral over the edges of  $T_n$  and then change the order of integration. This gives us the following integral ([5])

$$C_{mn} \int_{\partial T_n} (p - q) \times \mathbf{m}(\mathbf{r}') \cdot \int_{T_m} \frac{(\mathbf{r} - p)}{4\pi R} dS dl'.$$

Now the inner integral (with respect to  $\mathbf{r}$ ) is evaluated analytically and the outer integral is regular and allows numerical integration. Term  $I_3$  with an  $\mathbf{n} \times \text{RWG}$  testing function is considered by applying the same idea. More details are presented in [5].

## 4 EXAMPLES

In order to verify the stability of TENH/NETH formulation, we consider a dielectric sphere with  $\varepsilon_r = 4$ ,  $r = 1/k_0$ ,  $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$  illuminated by an axially incident plane wave. The figures on the LHS of Figure 1 show the equivalent electric and magnetic surface current densities on the surface of a sphere. The results show a good agreement with the present method and the other methods. Next we study the accuracy of the developed integration routines. We consider singular integral (5) with  $C_{mn} = 4\pi$  in the case where  $T_m$  and  $T_n$  share an edge. We see that the Gaussian quadrature without extracting the singularity leads to a significant error (solid line with circles on the RHS of Figure 1).

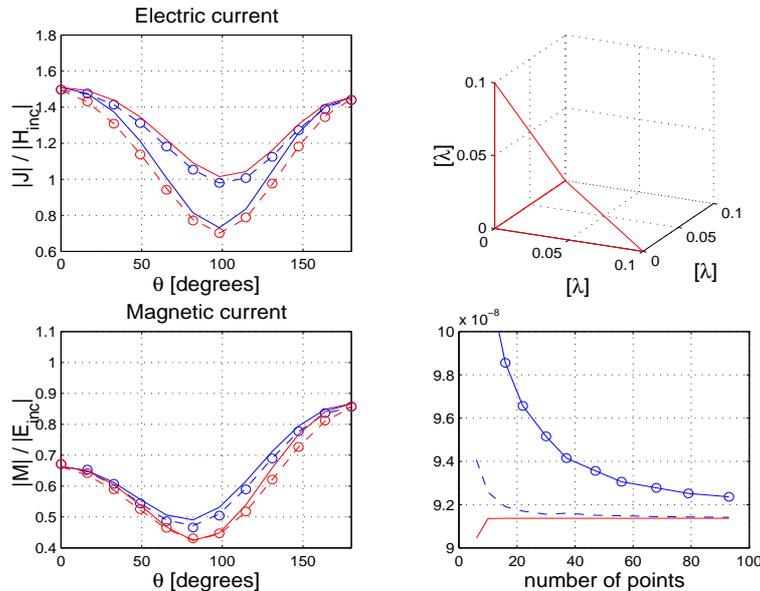


Fig. 1: On the LHS are the electric and magnetic current densities on a dielectric sphere. The figure displays CFIE (TENH/NETH) solution (solid line with circles), PMCHW solution (dashed line) and analytical solutions (solid line). On the horizontal axis is the angle from the axis of the sphere. The figures on the RHS show the geometry and the integral (5) in the case where triangles  $T_m$  and  $T_n$  have a common edge. The value of the integral is computed with three methods: the method of this paper (solid line), traditional singularity extraction and Gaussian (dashed line), double Gaussian without singularity extraction (solid line with circles). Horizontal axis shows the number of integration points.

## References

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