ON THE USE OF THE FRILL GENERATOR

IN INTEGRAL EQUATIONS FOR WIRE ANTENNAS

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ABSTRACT

For a frill-generator feed, certain fundamental mathematical properties of Hallén's and Pocklington's equations with the approximate kernel are developed. For a particular moment-method procedure (Galerkin's method with pulse functions applied to Hallén's equation), the consequences to the numerical solutions are carefully examined. Generalizations to other numerical methods with subsectional basis functions are discussed. Many of the results in this paper come from studying the simpler problem of the infinite dipole analytically and applying the understanding thus obtained to the case of the finite dipole.

INTRODUCTION

The simplest type of wire antenna or scatterer is the straight cylindrical dipole of length $2h$ and radius a. The integral equation satisfied by the current distribution on the dipole is usually referred to as Hallen's equation (HE). There are two choices of kernel, the exact and the approximate or reduced kernel. The approximate kernel is the main focus of this paper.

Extensions of HE, or of the corresponding integro-differential equation of the Pocklington type (PE), apply to complicated,"real-life" structures such as curved wire antennas and arrays of wire antennas. Such equations are usually dealt with by moment methods. Therefore, it is important to thoroughly understand the difficulties associated with the application of moment methods to HE/PE. (It is true that blind application of numerical methods can often give good results; at the very least, a thorough understanding of the difficulties helps one trust—or distrust—one's results). It is especially important to know the types of error that can occur, as well as the reason for the errors.

Both HE and PE have been around for many, many years; they have been dealt with by moment methods since the mid-sixties. Nonetheless, for the case of the approximate kernel, and for subsectional basis functions, an in-depth study of associated errors was only performed recently [1]. In [1] the dipole is center-driven by a delta-function generator. The most important difficulties are shown to be the appearance of rapid oscillations near the driving point when the number of basis functions becomes larger than h/a. The underlying reason for these difficulties is the fact that, with the delta-function generator, neither HE nor PE has a solution. This nonsolvability was known as far back as 1952, but is frequently not mentioned in recent textbooks. In [1], the basic tool of study is the idealized model of the infinite (in length) dipole, for which predictions can be made analytically. In [2], for the case of plane-wave incidence, it is shown that HE/PE for the finite dipole are, once again, nonsolvable when the approximate kernel is used. The consequence in this case is the appearance of oscillations near the ends of the dipole.

The purpose of the present paper is to extend the discussions of [1] and [2] for the case where the feed is a frill generator. We specifically refer to the "magnetic frill generator" described on p. 394 of [3]. The situation turns out to be quite different from the case of the delta-function generator.

BACKGROUND: EQUATIONS FOR FINITE DIPOLE

Adopting a notation similar to [1], we assume an $e^{-i\omega t}$ time dependence and consider a finite dipole parallel to the z-axis. We denote the approximate kernel by $K_{\mathrm{ap}}(z,a)$ and the unknown current by $I_{\mathrm{ap}}(z)$. For the exact kernel we denote the corresponding quantities by $K_{\mathrm{ex}}(z,a)$ and $I_{\mathrm{ex}}(z)$, whereas a symbol $K(z,a)$ or $I(z)$ with no subscripts can denote either quantity. PE is [3]

$$(\frac{\partial^2}{\partial z^2} + k^2) \int_{-h}^{h} K(z - z', a) I(z')\, dz' = g(z), \quad -h < z < h, \tag{1}$$

where the right-hand side (RHS) $g(z)$ can be conveniently written in terms of the approximate kernel:

$$g(z) = \frac{2\pi ikV}{\zeta_0 \ln(b/a)} \left[K_{\mathrm{ap}}(z, a) - K_{\mathrm{ap}}(z, b)\right]. \tag{2}$$

Here, V is the equivalent to the frill driving voltage, b is the radius of the outer conductor of the feeding coaxial transmission line, $\zeta_0 = 376.73\,\Omega$, and $k = \omega/c$. The corresponding form of HE is

$$\int_{-h}^{h} K(z - z', a) I(z')\, dz' = \frac{1}{k} \int_{0}^{z} g(t) \sin k(z - t)\, dt + C \cos kz, \quad -h < z < h, \tag{3}$$

where the constant C is to be determined from $I(\pm h) = 0$. Formulas for the kernels $K(z, a)$ can be found in [1]. Also contained in [1] is a detailed description of Galerkin's method with pulse functions, the numerical method to be used in this paper.

INFINITE DIPOLE

For the infinite dipole, PE is

$$(\frac{\partial^2}{\partial z^2} + k^2) \int_{-\infty}^{\infty} K(z - z', a) I^{(\infty)}(z')\, dz' = g(z), \quad -\infty < z < \infty. \tag{4}$$

To find the corresponding HE we note that: (i) even when $h = \infty$, (3) continues to hold; and that (ii) the RHS of (3) must represent an outgoing wave when $z \to \pm\infty$. This leads to the HE

$$\int_{-\infty}^{\infty} K(z - z', a) I^{(\infty)}(z')\, dz' = r(z), \quad -\infty < z < \infty, \tag{5}$$

$$r(z) = \frac{1}{2ik} e^{ikz} \int_{-\infty}^{z} g(t) e^{-ikt}\, dt + \frac{1}{2ik} e^{-ikz} \int_{z}^{\infty} g(t) e^{ikt}\, dt. \tag{6}$$

We have shown that the RHS $r(z)$ of HE behaves as follows for large z

$$r(z) \sim \frac{V}{2\zeta_0} e^{ik|z|}, \quad \text{as } z \to \pm\infty \qquad (\mathrm{Im}\{k\} \geq 0). \tag{7}$$

Thus, when the medium surrounding the dipole becomes slightly lossy, $r(z)$ is exponentially small for large distances. Note, incidentally, that the RHS of (7), which is independent of b and a, is precisely the RHS of HE for the case of the delta-function generator.

The integral in the left-hand side (LHS) of (5) is a convolution, so that (5) can be solved by Fourier transformation (FT). We initially assume that the medium is slighty lossy with $\mathrm{Im}\{k\} > 0$. Then, because of (7), $r(z)$ possesses an FT $\bar{r}(\zeta)$. Taking the FT of (5) immediately yields the FT of $I^{(\infty)}(z)$, and the inverse FT gives

$$I^{(\infty)}(z) = \frac{2ikV}{\zeta_0 \ln(b/a)} \int_{0}^{\infty} \frac{\left[\bar{K}_{\mathrm{ap}}(\zeta, a) - \bar{K}_{\mathrm{ap}}(\zeta, b)\right] \cos \zeta z}{(k^2 - \zeta^2) \bar{K}(\zeta, a)}\, d\zeta, \quad -\infty < z < \infty, \tag{8}$$

where (6) and (2) were used. We can now remove the assumption $\mathrm{Im}\{k\} > 0$ if the integration path in (8) passes below the point $\zeta = k$. For either kernel, eqn. (8) is the explicit solution to the HE (5) and, also, to the PE (4). For either kernel, the FT $\bar{K}$ involves Bessel functions, see [1].

The important point is that, even with the approximate kernel, the integral in (8) is convergent. This can be seen from the asymptotic behavior of the Bessel functions involved in (8). Thus, when

$K = K_{\rm ap}$ in (8), the denominator of the integrand is exponentially small, and so is the numerator. But since $b > a$, the entire integrand behaves like $\cos\zeta z/\zeta^2$ and the integral converges. This should be contrasted to the delta-function generator case, in which only the denominator is exponentially small, so that the integrand is exponentially large and the integral diverges. *For the infinite dipole, and with the approximate kernel, HE and PE are nonsolvable for the case of the delta-function generator. In contrast, they are solvable for the case of the frill generator and the solution is (8).*

One can apply our numerical method to the HE (5) for the infinite antenna, as done in [1] for the case of the delta-function generator. Since (5) is solvable, one should expect that the numerical solution thus obtained converges to the true solution (8) in the limit of zero pulse width. We have verified this expectation analytically. For the approximate kernel, the situation here is very different from the case [1] of the delta-function generator. There, the numerical solution diverges in the limit, and its imaginary part oscillates rapidly near the center $z = 0$ of the infinite dipole.

FOR THE FINITE DIPOLE, HE AND PE ARE NONSOLVABLE

We now show that, with the approximate kernel, (1) and (3) are *nonsolvable*. Assume there is a solution $I(z)$ to (1) that is continuous in $-h < z < h$. Now think of z as a complex variable and consider values of z not lying on the line segment $-h < z < h$. From [2], the LHS of (1) is an analytic function of the complex variable z with the possible exception of two line segments centered at $z = \pm ia$. Each segment has length $2h$ and is parallel to the real axis. By the theory of analytic continuation, the RHS $g(z)$ of (1) must also possess this analyticity property. But it follows from (2) and the formula [1] for the approximate kernel that $g(z)$ has branch points at $z = \pm ib$. Because these points cannot lie on the aforementioned two segments, we have a contradiction which shows that (1) cannot have a solution. Nonsolvability of (1), finally, also implies nonsolvability of (3).

In the literature, one finds many integral equations that use the approximate kernel. By now, it should be clear that "most" of them are nonsolvable. The integral equations (4) and (5) for an *infinite* dipole driven by a frill generator are therefore exceptional.

BEHAVIOR OF THE NUMERICAL SOLUTIONS

What do the numerical solutions for the infinite and finite dipoles have in common? The most notable common feature is the lack of oscillations near the driving point when the approximate kernel is used. We have verified this by numerical calculations. A specific example is provided when $h/\lambda = 0.25$, $a/\lambda = 0.007022$, $b/a = 2$, and $N = 200$ (compare with [1, Fig. 2] where corresponding results for the case of the delta-function generator are pictured). In our example, there are $2N+1 = 401$ pulse basis functions $u_n(z)$; $u_0(z)$ is centered at $z = 0$, and $u_N(z)$ corresponds to the endpoint $z = h$. Here, in contrast to the case of the delta-function generator, no oscillations occur near the driving point. Oscillations, however, do occur near the endpoint ($n = N = 200$).

The behavior just described is typical and, when N is large, *the oscillations near the endpoint are the main consequence of the nonsolvability of (3).* We have studied these oscillations through extensive numerical investigations and have determined the following:

(i) Suppose that N is varied from small to large values. For both the imaginary part $\mathrm{Im}\{I_{\mathrm{ap},n}/V\}$ and for the real part $\mathrm{Re}\{I_{\mathrm{ap},n}/V\}$, oscillations first appear when N becomes larger than the important parameter h/a. They become more rapid as N increases.

(ii) For fixed h/λ, a/λ, and N, the oscillating values change very little if b/λ is modified; they are numerically close to the corresponding values for the case of the delta-function generator.

For the delta-function generator case, the oscillations near $z = 0$ are shown in [1] not to be limited to our aforementioned numerical method. By analogy, oscillations similar to the ones described previously also occur, both in HE and in PE, when other subsectional basis and testing functions are used.

FINITE DIPOLE: MATRIX ILL-CONDITIONING

All numerical results discussed in the previous section have been carefully tested and are believed to be free of roundoff error. *The oscillations near $z = h$ are not due to roundoff error.* As a consequence, the oscillations do not depend on the particular hardware and software used, and they cannot be overcome by using more powerful computers.

In particular, the oscillations are not due to matrix ill-conditioning. However, as typically occurs in Fredholm integral equations of the first kind, matrix ill-conditioning is also an important effect. With numerical results, we have concluded that

(i) For sufficiently large N, and to an excellent degree of approximation, the condition number c

grows *exponentially* with the matrix size N.
(ii) For fixed large N, c is a rapidly increasing function of a/h. Furthermore, different values of h/λ and a/λ yield the same c as long as h/a is fixed. Thus h/a is, once again, the important parameter.

SUMMARY

For a dipole driven by a magnetic frill generator, and for HE/PE with the approximate kernel, the main conclusions in this paper are the following.

(i) For the infinite dipole, both HE and PE are solvable. (In contrast to the case [1] of the delta-function generator, where HE and PE are nonsolvable).

(ii) For the finite dipole, in contrast to the case of infinite length, HE and PE are nonsolvable.

(iii) For the infinite dipole, the numerical solution obtained by applying a particular numerical method converges to the true solution in the limit of basis functions of zero width. (For the case [1] of the delta-function generator, the numerical solution diverges in this limit, and rapid oscillations occur near the driving point; very similar oscillations also occur when the dipole is finite).

(iv) An important common feature between the finite and infinite dipoles is that, for large N, no oscillations occur near their driving points. This was verified by numerical results.

(v) With numerical results, it was demonstrated that oscillations do occur near the ends of the finite dipole. The important parameter here was found to be the number of points divided by h/a, not the number of points per wavelength as one might expect. These oscillations are the main consequence of the nonsolvability of HE/PE.

(vi) The aforementioned parameter was also shown to be closely related to matrix condition numbers, and the separate—but very important—effect of matrix ill-conditioning was described.

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