A MULTI-RESOLUTION HOMOGENIZATION (MRH) ANALYSIS FOR THE FIELDS NEAR THE INTERFACE BETWEEN COMPLEX LAMINATES

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ABSTRACT

The multiresolution homogenization theory (MRH) for EM fields in multi-scale laminates is extended here to address lateral interfaces between complex laminates. The scattered field is found by matching the spectra (generally discrete and continuous) on both sides of the interface. Special attention is given to the boundary layer (a surface state) field near the interface, which is described by higher order evanescent modes that are generated by the micro-scale heterogeneities. Far from the interface the field is described asymptotically by effective vertical modes whose horizontal propagation is described in terms of rays that refract at the interface via an effective Snell's law.

INTRODUCTION

Multi-resolution homogenization (MRH) is a systematic theory for deriving effective macro-scale formulations for source-excited electromagnetic fields in complex laminates. It smoothes out the micro-scale heterogeneities while retaining their effect on the macro scale (physical) observables [1]–[3]. The theory does not rely on local periodicities or on weak perturbation. It also can handle scale continuum and therefore is amenable for treating ultra wideband (UWB) fields [4].

The MRH has been derived originally for strictly stratified media, but it is being extended here to complex laminates configurations with lateral variations. For configurations with smooth variation, we have previously formulated a theory of "vertical effective modes and horizontal rays" [5], wherein the horizontal (lateral) propagation of the effective vertical modes is described by a two dimensional modal-ray equation that accounts for the lateral variations of the medium as expressed by the effective measures of the complex vertical stratification. In the present paper we explore the fields in the vicinity of a lateral interface separating two complex laminates as depicted in Figure 1. Using the rigorous construct of the MRH, the source-excited field on each side of the interface is described by the corresponding effective spectrum (generally consisting of both discrete and continuous spectra) and the scattered field is then determined by matching the spectra on the two sides. Special attention is given to the boundary layer field near the interface, which is described by higher order evanescent modes that are generated by the micro-scale heterogeneities and therefore generates a surface state. Far from the interface the fields is reduced asymptotically to a sum od effective vertical modes whose horizontal propagation is described in terms rays that are refracted at the interface via an effective Snell's law.

FORMULATION OF THE PROBLEM

To explore the interface problem we consider the simplest configuration where the medium on one side of the interface ($x<0$) is a homogenous slab while on the other side ($x>0$) the medium is a complex laminate (see Fig. 1). The coordinate along the stratification axis is taken to be z, while the transversal coordinates are $\boldsymbol{\rho}=(x,y)$. Both slabs reside in $0<z<d$ and are surrounded by free space at $z<0$ and $z>d$. The relative constitutive parameters of the homogeneous slab are ε_1, μ_1 while in the complex laminate, they are diagonal tensors whose components $\varepsilon_z, \varepsilon_t, \mu_z, \mu_t$ (the subscripts z and t denote the z and transverse components, respectively) are *multi-scale* functions of z, comprising of both macro and micro scales.

To demonstrate the theory for the 3D electromagnetic field in a simple format we consider an excitation by a vertically polarized current dipole $\mathbf{J}=\hat{\mathbf{z}}J_0\delta(\mathbf{r}-\mathbf{r}')$ residing at $\mathbf{r}'=(x',y',z')$, $x'<0$ in the uniform medium. The resulting transverse magnetic (TM) field is fully described by the scalar potential that satisfies

$$\Big[\frac{\partial^2}{\partial z^2}+\nabla_t^2+k_0^2\varepsilon_1\mu_1\Big]G^{(1)}(\mathbf{r},\mathbf{r}')=-\varepsilon_1 J_0\delta(\mathbf{r}-\mathbf{r}'),\quad x<0 \tag{1a}$$

$$\Big[\frac{\partial}{\partial z}\frac{1}{\varepsilon_t(z)}\frac{\partial}{\partial z}+\frac{1}{\varepsilon_z(z)}\nabla_t^2+k_0^2\mu_t(z)\Big]G^{(2)}(\mathbf{r},\mathbf{r}')=0,\quad x>0 \tag{1b}$$

with the continuity condition $G^{(1)}|_{0-}=G^{(2)}|_{0+}$ and $\varepsilon_1^{-1}\partial_x G|_{0-}=\varepsilon_z^{-1}(z)\partial_x G|_{0+}$ at the interface $x=0$. Here $\nabla_t^2=\partial_x^2+\partial_y^2$ and the superscripts $(1,2)$ correspond to the solution in the region $x\lessgtr 0$, respectively. Once G is found and properly parameterized, the electromagnetic field can be readily found by applying certain ∇ operations (see e.g. [6]).

The incident field $G_i^{(1)}$ near the interface can be described by a summation over the propagating modal functions $G_i^{(1)}(\mathbf{r},\mathbf{r}') = \sum_n U_n^{(1)}(z)U_n^{(1)}(z')G_{2D}^{(1)}(\boldsymbol{\rho},\boldsymbol{\rho}';k_{t\,n}^{(1)})$ where $U_n^{(1)}(z)$ are the modes in the uniform slab. The $\sum_n$ sign is used symbolically and denotes both discrete and continuous spectral constituents. $G_{2D}^{(1)}(\boldsymbol{\rho},\boldsymbol{\rho}';k_{t\,n}^{(1)}) = \frac{i}{4}H_0^{(1)}(k_{t\,n}^{(1)}|\boldsymbol{\rho}-\boldsymbol{\rho}'|)$ is the transversal domain Green's function for nth mode with $k_{t\,n}^{(1)}$ being the modal transversal wavenumber and $H_0^{(1)}$ is the Hankel function of the first kind.

EFFECTIVE FORMULATION IN THE MULTISCALE LAMINATE

We start by summarizing the relevant concepts of the MRH theory for multi-scale heterogeneous laminates with uniform cross section [2,3]. In the effective formulation one is mainly concerned with the large-scale components of the response. The effective formulation is therefore checked and validated on the *field homogenization scale* (F-HS) which is defined on a binary scale to be F-HS$=2^{-j}$. In most cases the F-HS it is determined by the *local wavenumber* in the *effective medium* to be $(k_0\sqrt{\varepsilon_{\max}\mu_{\max}})^{-1}$. In certain cases, however, the wave-physics may suggest a *larger* F-HS (e.g. if the sources excite only the N lowest order modes that propagate transversely in the slab, then their z-domain scale is given essentially by $d/2\pi N$ and the F-HS can be taken to be much larger than $(k_0\sqrt{\varepsilon_{\max}\mu_{\max}})^{-1}$). The F-HS may also be determined by the detector size. The homogenization (or observables) space $\mathbb{V}_j$ is defined now as the resolution space [7] that contains all scales larger or equal to the F-HS.

The medium is described on a different scale, termed the *medium homogenization scale* M-HS$=2^{-\nu}$, and the error of the effective formulation (see (3)) depends on the scale $2^{-(\nu+1)}$ which is the maximal scale that is neglected in the homogenization. Actually, if the medium has a scale gap then the homogenized medium is described on a larger scale, say $=2^{-M}$ with $M<\nu$, yet the error still depends on the neglected scale $2^{-(\nu+1)}$. Thus for simplicity we refer only to the general case where M-HS$=2^{-\nu}$. The *effective medium* is then obtained by projecting the actual medium parameters $\varepsilon_{z,t}$ and $\mu_{z,t}$ onto the smooth resolution space defined by the M-HS via

$$\varepsilon_t^{\mathrm{ef}} = \mathcal{P}_\nu\varepsilon_t, \quad \varepsilon_z^{\mathrm{ef}} = [\mathcal{P}_\nu(\varepsilon_z^{-1})]^{-1}, \quad \mu_t^{\mathrm{ef}} = \mathcal{P}_\nu\mu_t, \quad \mu_z^{\mathrm{ef}} = [\mathcal{P}_\nu(\mu_z^{-1})]^{-1}. \tag{2}$$

where $\mathcal{P}_\nu$ is the projection operator (see [7]).

Expanding the source excited field into a plane wave spectrum in the (x,y) domain (Fourier transform) with $k_t = \sqrt{k_x^2+k_y^2}$, it was proven in [2] that the 1D source excited spectral solutions $U(z;k_t)$ in the actual and in the effective media are equivalent with an accuracy bounds given by

$$\frac{\left\|\mathcal{P}_jU - \mathcal{P}_jU^{\mathrm{ef}}\right\|}{\left\|\mathcal{P}_jU^{\mathrm{ef}}\right\|} \sim k_0Kd\,N_\nu^2M_\nu^2 \tag{3}$$

In view of the $\mathcal{P}_j$ projection, the left hand side in (3) represents the error in the homogenization space $\mathbb{V}_j$ defined above.

The parameter $M_\nu = \left\|\varepsilon_t^d\right\|^2/\left\|\varepsilon_t^{\mathrm{ef}}\right\|^2 + \left\|p^d\right\|^2/\left\|p^{\mathrm{ef}}\right\|^2$ with $p = \mu_t\varepsilon_z^{-1} - k_t^2$ defines the ratio between the norm of the detail and smooth parts of the medium heterogeneities. Here the superscript ef defines the smooth part of a function (e.g., $p^{\mathrm{ef}} = \mathcal{P}_\nu p^{\mathrm{ef}}$) while the superscript d denote the detail components which are left out. $K=\max_z\{\sqrt{p^{\mathrm{ef}}\varepsilon_t^{\mathrm{ef}}}\}$ is a bound on the normalized z directed effective wavenumber. The error in (5) is controlled by the *dimensionless* scale-parameter N_ν defined via $N_\nu = k_0K2^{-\nu}$. Recalling the consideration above, concerning the choice of the F-HS, $N_\nu \leq \frac{M-HS}{F-HS}$, hence it can be used as a built-in parameter controlling the error, leading to the fact that the MRH is not a perturbation theory in M_ν, and it does not require a scale-gap in the medium.

As noted after (1), the field in the laminate configuration will be expanded in terms of the spectrum in z direction that is given by the eigenvalue associated with equation (1b)

$$\Big[\frac{d}{dz}\frac{1}{\varepsilon_t}\frac{d}{dz} + k_0^2\mu_t + \varepsilon_z^{-1}\lambda_n\Big]U_n(z) = 0 \tag{4}$$

where U_n and λ_n are the eigenfunctions and eigenvalues of the actual problem which are assumed to be normalized with respect to the weight function ε_z^{-1}, i.e. $\left\langle U_m, \varepsilon_z^{-1}U_n\right\rangle = \delta_{mn}$. If the slab is closed in the z direction with perfectly conducting or impedance boundaries, then there are only discrete eigenvalues for $\lambda_n \in [-k_0^2\epsilon_{\max}\mu_{\max},\infty)$, but if it is open then they are discrete for $\lambda_l \in [-k_0^2\epsilon_{\max}\mu_{\max}, -k_0^2)$ and continuous for $\lambda \in [-k_0^2,\infty)$. The effective modes are governed by (4) but with the medium parameters replaced with the effective parameters in (2).

In [3] we proved that the error bound for the eigenfunctions is the same as that in (3) for the source excited solutions. The error bound for the eigenvalues is given by

$$\left|\lambda_n - \lambda_n^{\mathrm{ef}}\right| \sim k_0^2K^2N_\nu^2M_\nu^2 \tag{5}$$

where all parameters are defined in (3). As follows from the bounds in (3)–(5) choosing some j and ν we obtain the spectral parameter λ_j that defines the highest mode n_j (or the maximal entry in the continuous spectrum) for which the effective formulation is valid.

EFFECTIVE FORMULATION FOR THE INTERFACE

In order to calculate the field in Equation (1) we expand $G^{(1,2)}$ in terms of the corresponding eigenfunctions $U_n^{(1)}(z)$ and $U_n^{(2)}(z)$ that propagate horizontally: The y propagation is represented as the Fourier transform over the spectral variable k_y while the x propagation is given in terms of a sum of the incident, reflected and transmitted mode for each index n and each spectral variable k_y. Therefore the scalar potential for $x > x'$ is given by

$$G^{(1)} = \frac{1}{2\pi}\sum_l U_l^{(1)}(z')\int dk_y(-2ik_{x\,l}^{(1)})^{-1}e^{-ik_{x\,l}^{(1)}x'}\left[e^{i(k_{x\,l}^{(1)}x+k_y y)}U_l^{(1)}(z)+\sum_n \Gamma_n^l e^{i(-k_{x\,n}^{(1)}x+k_y y)}U_n^{(1)}(z)\right] \tag{6a}$$

$$G^{(2)} = \frac{1}{2\pi}\sum_l U_l^{(1)}(z')\int dk_y(-2ik_{x\,l}^{(1)})^{-1}e^{-ik_{x\,l}^{(1)}x'}\sum_n T_n^l e^{i(k_{x\,n}^{(2)}x+k_y y)}U_n^{(2)}(z) \tag{6b}$$

where l denotes the index of the incident modes while n is the index of the reflected and transmitted modes; the n and l summations may consist of both discrete and continuous spectral constituents. Here $k_{x\,n}^{(1,2)}$ and k_y are related via the dispersion relation $\left(k_{x\,n}^{(1,2)}\right)^2 + k_y^2 = \left(k_{t\,n}^{(1,2)}\right)^2$ where the horizontal wave numbers $k_{t\,n}$ are implied by the modal parameter via $k_{t\,n}^{(1,2)} = \sqrt{-\lambda_n^{(1,2)}}$. We have $k_t > 0$ for $\lambda < 0$ (propagating spectrum) and $k_t = i\sqrt{\lambda}$ for $\lambda > 0$ (evanescent spectrum).

The first term in (6a) is the plane wave expansion of the incident field (see discussion after (1)). The coefficients Γ_n^l and T_n^l correspond to the nth reflected and transmitted modes due to the l incident mode and together they form the *scattering matrix*. The eigenvalues $\lambda_n^{(1)}$ and eigenfunctions $U_n^{(1)}$ in the uniform slab are known exactly. In the heterogeneous slab, the first n_j modes may be described by the effective modes $U_n^{(2)\,\mathrm{ef}}(z)$, but as will be discussed after (8), we must retain also some of the higher order modes that have finer details than the F-HS and therefore are not described by the effective theory and are calculated using the actual heterogeneous medium.

Field evaluation far from the interface. In the far zone the integrals in (6) can be evaluated asymptotically and the field can be represented as a sum of reflected and refracted horizontal modal rays

$$G^{(1)} = \sum_l U_l^{(1)}(z')\frac{e^{i\pi/4}}{\sqrt{8\pi k_{t\,l}^{(1)}}}\left[\frac{e^{ik_{t\,l}^{(1)}|\boldsymbol{\rho}-\boldsymbol{\rho}'|}}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}}+\sum_n \Gamma_n^l\frac{e^{i[k_{t\,l}^{(1)}L_l^{(0)}+k_{t\,n}^{(1)}L_n^{(1)}]}}{\sqrt{L_l^{(0)}+L_n^{(1)}\frac{k_{t\,l}^{(1)}\cos^2\theta_l^{(1)}}{k_{t\,n}^{(1)}\cos^2\theta_n^{(1)}}}}\right]U_n^{(1)}(z) \tag{7a}$$

$$G^{(2)} = \sum_l U_l^{(1)}(z')\frac{e^{i\pi/4}}{\sqrt{8\pi k_{t\,l}^{(1)}}}\sum_n T_n^l\frac{e^{i[k_{t\,l}^{(1)}L_l^{(0)}+k_{t\,n}^{(2)}L_n^{(2)}]}}{\sqrt{L_l^{(0)}+L_n^{(2)}\frac{k_{t\,l}^{(1)}\cos^2\theta_l^{(1)}}{k_{t\,n}^{(2)}\cos^2\theta_n^{(2)}}}}U_n^{(2)}(z) \tag{7b}$$

where $\theta_l^{(1)}$, $\theta_n^{(1)}$ and $\theta_n^{(2)}$ are the propagation angles of the incident, reflected and transmitted horizontal modal rays (see Fig. 2). They are found via Snell's law for these rays: $k_{t\,l}^{(1)}\sin\theta_l^{(1)} = k_{t\,n}^{(1)}\sin\theta_n^{(1)} = k_{t\,n}^{(2)}\sin\theta_n^{(2)}$. The distances $L_l^{(0)} = |x-x'|/\cos\theta_l^{(0)}$, $L_n^{(1)} = |x'|/\cos\theta_n^{(1)}$ and $L_n^{(2)} = |x|/\cos\theta_n^{(2)}$ are defined in Fig. 2.

Note that the summation in (7) comprises only of the propagating (lower order) modes that are well described by the effective theory. Thus one can replace $U_n^{(2)}$, $\xi_n^{(2)}$ and $\theta_n^{(2)}$ with $U_n^{(2)\mathrm{ef}}$, $\xi_n^{(2)\mathrm{ef}}$ and $\theta_n^{(2)\mathrm{ef}}$, i.e. the far zone field propagates along the effective modal rays defined by the effective angles of propagation as depicted in Fig. 2. However, as will be shown next, the excitation coefficients of this rays can be affected not only by the lower order modes but also by the higher order modes that have to be calculated in the actual heterogeneous slab.

Calculation of the scattering coefficients Γ_n^l and T_n^l: We apply the continuity conditions for G and for $\varepsilon_z^{-1}\partial_x G$ in (1) and then perform the inner product of the resulting equations with $\varepsilon_z^{-1}U_m^{(2)}$ and $U_m^{(1)}$ ($m = 1, 2\cdots$), respectively, utilizing the orthogonality condition stated after (4). This leads to the matrix equation

$$\mathbf{D}(\mathbf{I}+\boldsymbol{\Gamma}) = \mathbf{T} \tag{8a}$$

$$\mathbf{K}^{(1)}(\mathbf{I}-\boldsymbol{\Gamma}) = \mathbf{D}^t\mathbf{K}^{(2)}\mathbf{T} \tag{8b}$$

Here $\mathbf{\Gamma} = \Gamma_n^l$ and $\mathbf{T} = T_n^l$ are cross-mode reflection and transmission matrices. $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ are diagonal matrices whose elements are given by $K_{nn}^{(1)} = k_{x\,n}^{(1)}$ and $K_{nn}^{(2)} = k_{x\,n}^{(2)}$. $\mathbf{D}$ is a cross-coupling matrix whose elements are defined as $D_{mn} = \left\langle \varepsilon_z^{-1} U_m^{(2)}, U_n^{(1)} \right\rangle$. This matrix is of great importance since it defines the cross-coupling of the lower order and higher order modes, i.e. the excitation of the higher order modes and their effect on the lower order modes.

The elements of this matrix have different properties in the following two cases.

***(i)* The weighting function ε_z is uniform (say $\varepsilon_z = 1$):** In this case the elements of $\mathbf{D}$ are given by $D_{mn} = \varepsilon_1^{-1} \left\langle U_m^{(2)}, U_n^{(1)} \right\rangle$. In view of the fact that the higher order modes are fast varying we find that they are non-negligible mainly for $m \sim n$, hence the coupling between the lowest order incident modes and the higher order modes is weak. Thus the coefficient corresponding to the lower order modes are affected only by the effective modes, and the higher order modes are not included in the calculations. Also the excitation of these modes (the surface state) is very weak.

***(ii)* The weighting function ε_z^{-1} is heterogeneous:** In this case there may be a strong coupling between the lower order and the higher order modes. To demonstrate this phenomenon we consider a simple case where the micro-heterogeneity consists only of one micro scale $2^{-\nu}$ and there is a large gap between this scale and the macro scales. In this case the coefficients of the cross-coupling matrix $\mathbf{D}$ corresponding to the modes with variation scale $2^{-\nu}$ may be large leading to excitation of the higher order modes with index close to the micro structure of the medium. These modes are typically beyond the cutoff and are evanescent in the x direction, hence they generate the *surface state* that resides near the interface.

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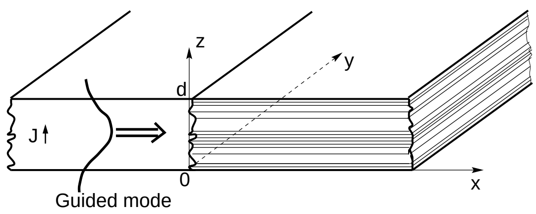

Figure 1: Problem formulation.

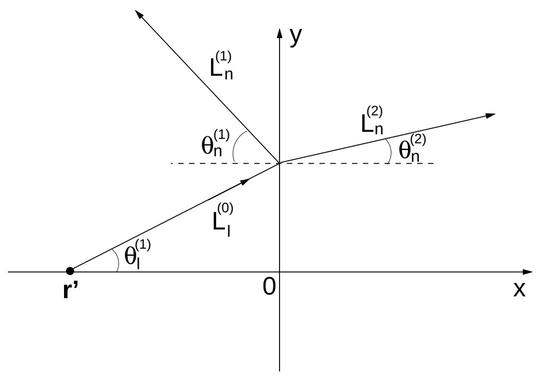

Figure 2: Reflection and refraction of the horizontal modal rays.