GALERKIN METHOD FOR SOLVING OF SINGULAR INTEGRAL EQUATION OF DIFFRACTION PROBLEM [1]

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ABSTRACT

The problem of diffraction of electromagnetic waves in a parallelepiped resonator is considered. The method of volume integral singular equations is applied for solving of the problem. The solution of the stated problem is expressed in terms of vector potentials. Using the tensor Green function of parallelepiped new integral equations are presented. Galerkin method is used for numerical solution. The approximate solution is found in the form of linear combination of basis "hat"-functions. The detailed algorithm of construction of basis functions is given. Explicit formulas for the elements of augmented matrix of Galerkin method are presented.

THE STATEMENT OF THE DIFFRACTION PROBLEM

Let $P = \{x : 0 \le x_1 \le a, 0 \le x_2 \le b, 0 \le x_3 \le c\}$ be a resonator with perfectly conducting boundary. Let Q be a three-dimensional body, located in P. Q is characterized by tensor permittivity $\hat{\varepsilon}$ and constant permeability μ_0. We suppose that components of $\hat{\varepsilon}$ are smooth functions in $\bar{Q}$ and $\left(\frac{\hat{\varepsilon}}{\varepsilon_0} - \hat{I}\right)$ is invertible in $\bar{Q}$; $Q \cap \partial P = \emptyset$. Let $P/\bar{Q}$ be homogeneous and isotropic medium. Incident and diffraction fields depend on time variable as $e^{-i\omega t}$.

We will find electromagnetic diffraction fields E and H, satisfying Maxwell's equations in $P \setminus \partial Q$:

$$\begin{array}{l} rot\ \vec{H} = -i\omega\hat{\varepsilon}\vec{E} + \vec{j}_E^{\,0} \\ rot\ \vec{E} = i\omega\mu\vec{H} - \vec{j}_H^{\,0} \quad . \end{array} \tag{1}$$

The complete field should have continuous tangent components at ∂Q :

$$\left[\vec{n} \times \vec{E}^c\right]\Big|_{\partial Q} = \left[\vec{n} \times \vec{H}^c\right]\Big|_{\partial Q} = 0$$

and must satisfy the following boundary condition:

$$\vec{E}^c_\tau|_{\partial P} = 0\ . \tag{2}$$

INTEGRO-DIFFERENTIAL EQUATIONS FOR THE DIFFRACTION PROBLEM

We will express the solution of the stated problem in terms of vector potentials $\vec{A}_E$ and $\vec{A}_H$ [3]:

$$\begin{array}{l} \vec{A}_E = \int\limits_Q \hat{G}_E(x,y)\vec{j}_E(y)dy\ , \vec{A}_H = \int\limits_Q \hat{G}_H(x,y)\vec{j}_H(y)dy\ , \\ \vec{E} = i\omega\mu_0\vec{A}_E - \frac{1}{i\omega\,\varepsilon_0} grad\ div\ \vec{A}_E - rot\ \vec{A}_H\ , \\ \vec{H} = i\omega\,\varepsilon_0\,\vec{A}_H - \frac{1}{i\omega\mu_0} grad\ div\ \vec{A}_H + rot\ \vec{A}_E\ . \end{array} \tag{3}$$

Here $\vec{j}_E = \vec{j}_E^{\,0} + \vec{j}_E^{\,p}$, $\vec{j}_H = \vec{j}_H^{\,0} + \vec{j}_H^{\,p}$, ($\vec{j}_E^{\,p}$, $\vec{j}_H^{\,p}$ are polarization currents). $\hat{G}_E, \hat{G}_H$ are Green functions for Helmholtz equation, conforming to the arbitrary currents $\vec{j}_E^{\,0}, \vec{j}_H^{\,0}$.

[1] Supported by Russian Foundation for Basic Research, grant 01–01–00053

$\hat{G}_E, \hat{G}_H$ are known [2] to have the form of diagonal tensors (the components of $\hat{G}_E$ are written out below):

$$G_E^1 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{2\,\varepsilon_n}{ab\gamma \mathrm{sh}\gamma c} \cos(\tfrac{\pi n}{a}x_1)\sin(\tfrac{\pi m}{b}x_2)\cos(\tfrac{\pi n}{a}y_1)\sin(\tfrac{\pi m}{b}y_2)\begin{cases}\mathrm{sh}\gamma \mathrm{x}_3\ \mathrm{sh}\gamma(\mathrm{c}-\mathrm{y}_3),\ \mathrm{x}_3<\mathrm{y}_3\\ \mathrm{sh}\gamma \mathrm{y}_3\ \mathrm{sh}\gamma(\mathrm{c}-\mathrm{x}_3),\ \mathrm{x}_3>\mathrm{y}_3\end{cases}$$

$$G_E^2 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{2\,\varepsilon_m}{ab\gamma \mathrm{sh}\gamma c} \sin(\tfrac{\pi n}{a}x_1)\cos(\tfrac{\pi m}{b}x_2)\sin(\tfrac{\pi n}{a}y_1)\cos(\tfrac{\pi m}{b}y_2)\begin{cases}\mathrm{sh}\gamma \mathrm{x}_3\ \mathrm{sh}\gamma(\mathrm{c}-\mathrm{y}_3),\ \mathrm{x}_3<\mathrm{y}_3\\ \mathrm{sh}\gamma \mathrm{y}_3\ \mathrm{sh}\gamma(\mathrm{c}-\mathrm{x}_3),\ \mathrm{x}_3>\mathrm{y}_3\end{cases} \quad (4)$$

$$G_E^3 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4}{ab\gamma \mathrm{sh}\gamma c} \sin(\tfrac{\pi n}{a}x_1)\sin(\tfrac{\pi m}{b}x_2)\sin(\tfrac{\pi n}{a}y_1)\sin(\tfrac{\pi m}{b}y_2)\begin{cases}\mathrm{ch}\gamma \mathrm{x}_3\ \mathrm{ch}\gamma(\mathrm{c}-\mathrm{y}_3),\ \mathrm{x}_3<\mathrm{y}_3\\ \mathrm{ch}\gamma \mathrm{y}_3\ \mathrm{ch}\gamma(\mathrm{c}-\mathrm{x}_3),\ \mathrm{x}_3>\mathrm{y}_3\end{cases}$$

Here $\gamma = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2 - k_0^2}$ (the proper branch for square root is chosen as in [1], §2.3), $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n = 1, 2, 3, \dots$.

We obtain the following integro-differential equations (under the condition $\hat{\mu} = \mu_0 \hat{I}$ in P):

$$\vec{E}(x) = \vec{E}^0(x) + k_0^2 \int_Q \hat{G}_E \left[\frac{\hat{\varepsilon}(y)}{\varepsilon_0} - \hat{I}\right]\vec{E}(y)dy + grad\ div \int_Q \hat{G}_E\left[\frac{\hat{\varepsilon}(y)}{\varepsilon_0} - \hat{I}\right]\vec{E}(y)dy\ ,$$

and we have (5)

$$\vec{H}(x) = \vec{H}^0(x) - i\omega\varepsilon_o\, rot \int_Q \hat{G}_E\left[\frac{\hat{\varepsilon}(y)}{\varepsilon_0} - \hat{I}\right]\vec{E}(y)dy,\ x \in Q.$$

GALERKIN METHOD

Let us introduce the following auxilary function

$$\tilde{G}(x,y) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4}{ab\gamma \mathrm{sh}\gamma c}\sin(\tfrac{\pi n}{a}x_1)\sin(\tfrac{\pi m}{b}x_2)\sin(\tfrac{\pi n}{a}y_1)\sin(\tfrac{\pi m}{b}y_2)\times$$
$$\times\begin{cases}\mathrm{sh}\gamma \mathrm{x}_3\ \mathrm{sh}\gamma(\mathrm{c}-\mathrm{y}_3),\ \mathrm{x}_3<\mathrm{y}_3\\ \mathrm{sh}\gamma \mathrm{y}_3\ \mathrm{sh}\gamma(\mathrm{c}-\mathrm{x}_3),\ \mathrm{x}_3>\mathrm{y}_3\end{cases}. \quad (6)$$

The derivatives of $\tilde{G}$ are connected to the derivatives of G_E^i through the equalities:

$$\frac{\partial G_E^i}{\partial x_i} = \frac{\partial \tilde{G}}{\partial y_i}, \quad i = 1, 2, 3\,. \quad (7)$$

We extract singularity of Green function $\hat{G}$. Using Fourier transformation and interpolation polynomials we obtain:

$$\hat{G}_E(x,y) = \frac{1}{4\pi}\frac{e^{ik_0|x-y|}}{|x-y|}\cdot\hat{I} + diag\{g_1(x,y), g_2(x,y), g_3(x,y)\},$$

where g_k are smooth functions.

Before describing the method itself we should make some transformations of equation (5). Denoting $\left(\frac{\hat{\varepsilon}(x)}{\varepsilon_0} - \hat{I}\right)^{-1}$ as $\hat{\xi}$ and $\left(\frac{\hat{\varepsilon}(x)}{\varepsilon_0} - \hat{I}\right)\vec{E}$ as $\vec{J}$ we obtain the following equation

$$A\vec{J} := \hat{\xi}\vec{J}(x) - k_0^2\int_Q \hat{G}_E\vec{J}(y)dy - grad\ div \int_Q \hat{G}_E\vec{J}(y)dy = \vec{E}_0(x) \quad (8)$$

Now let us write (8) as a system of three scalar equations:

$$\sum_{i=1}^{3}\xi_{li}J^i(x) - k_0^2\int_Q G_E^l(x,y)J^l(y)dy - \frac{\partial}{\partial x_l}div_x\int_Q \hat{G}(x,y)\vec{J}(y)dy = E_0^l(x), \quad l = 1, 2, 3. \quad (9)$$

We will determine the components of approximate solution $\vec{\bar{J}}$ in the following way:

$$\bar{J}^1 = \sum_{k=1}^{N} a_k f_k^1(x),\quad \bar{J}^2 = \sum_{k=1}^{N} b_k f_k^2(x),\quad \bar{J}^3 = \sum_{k=1}^{N} c_k f_k^3(x), \quad (10)$$

where f_k^i are basis "hat"-functions which depend essentially on x^i. The explicit form of f_k^1 is given below.

Let Q be a parallelepiped: $Q = \{x : a_1 \le x^1 \le a_2, b_1 \le x^2 \le b_2, c_1 \le x^3 \le c_2\}$, $Q \subset P$. We will cover Q with small parallelepipeds

$$\Pi^1_{klm} = \{x : x^1_{k-1} \le x^1 \le x^1_{k+1},\ x^2_l \le x^2 \le x^2_{l+1},\ x^3_m \le x^3 \le x^3_{m+1}\}$$
$$x^1_k = a_1 + \frac{a_2 - a_1}{n}k,\ x^2_l = b_1 + 2\frac{b_2 - b_1}{n}l,\ x^3_m = c_1 + 2\frac{c_2 - c_1}{n}m; \tag{11}$$

where $k = 1, \dots, n-1;\quad l, m = 0, 1, \dots, \frac{n}{2} - 1$.

Denoting $(x_k - x_{k-1})$ as h^1 we get the formulas for f^1_{klm} :

$$f^1_{klm} = \begin{cases} \frac{x^1 - x^1_{k-1}}{x^1_k - x^1_{k-1}},\ if\ x^1 \in [x^1_{k-1}; x^1_k]\ and\ x \in \Pi^1_{klm} \\ \\ \frac{x^1_{k+1} - x^1}{x^1_{k+1} - x^1_k},\ if\ x^1 \in [x^1_k; x^1_{k+1}]\ and\ x \in \Pi^1_{klm} \\ \\ 0, \qquad if\ x \notin \Pi^1_{klm} \end{cases} \tag{12}$$

or

$$f^1_{klm} = \begin{cases} 1 - \frac{1}{h^1}|x^1 - x^1_k|,\ if\ x \in \Pi^1_{klm} \\ 0, \qquad if\ x \notin \Pi^1_{klm} \end{cases}. \tag{13}$$

Functions f^2_{klm} and f^3_{klm} should be determined by similar formulas. Since

$$f^1_{klm}|_{x^1 \in \{x^1_{k-1}, x^1_{k+1}\}} = 0,\ f^2_{klm}|_{x^2 \in \{x^2_{l-1}, x^2_{l+1}\}} = 0,\ f^3_{klm}|_{x^3 \in \{x^3_{m-1}, x^3_{m+1}\}} = 0\ , \tag{14}$$

every component of approximate vector solution vanishes at some side of Q. However the constructed set of basis functions does satisfy the necessary approximation condition.

Introducing total enumeration for basis functions we get

$$f^1_k, f^2_k, f^3_k; \quad k = 1, \dots, N,$$

where $N = \frac{1}{4}(n^3 - n^2)$.

According to Galerkin method we can represent the augmented matrix for determining unknown coefficients a_k, b_k, c_k in the following form:

$$\left(\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & B_1 \\ A_{21} & A_{22} & A_{23} & B_1 \\ A_{31} & A_{32} & A_{33} & B_1 \end{array} \right) \tag{15}$$

where columns B_k and matrices A_{kl} are determined by formulas:

$$B^i_k = (E^k_0, f^k_i); \tag{16}$$

$$A^{ij}_{kl} = (\xi_{kl} f^l_j, f^k_i) - \delta_{kl} k_0^2 \left(\int_Q G^k_E(x,y) f^l_j(y) dy, f^k_i(x) \right) - $$
$$- \left(\frac{\partial}{\partial x_k} \int_Q \frac{\partial}{\partial x_l} G^k_E(x,y) f^l_j(y) dy, f^k_i(x) \right), \tag{17}$$

$k = 1, 2, 3;\quad i = 1, \dots, N$. (f,g) determines the scalar product in L_2, $(f, g) = \int_Q f(x) g(x) dx$.

Applying the formulas of integration by parts to both internal and external integrals and taking into account (7) and (14) we obtain:

$$A^{ij}_{kl} = (\xi_{kl} f^l_j, f^k_i) - \delta_{kl} k_0^2 \left(\int_Q G^k_E(x,y) f^l_j(y) dy, f^k_i(x) \right) - $$
$$- \left(\int_Q \tilde{G}(x,y) \frac{\partial}{\partial x_l} f^l_j(y) dy, \frac{\partial}{\partial x_k} f^k_i(x) \right). \tag{18}$$

NUMERICAL RESULTS

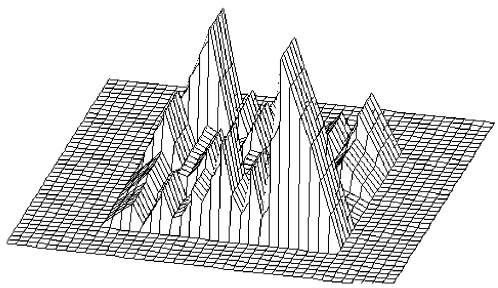
Let P and Q be parallelepipeds:

$$P = \{x : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1\}$$

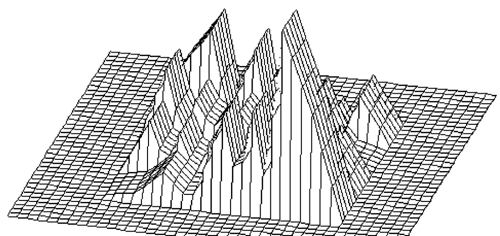
$$Q = \{x : 0.2 \leq x_1 \leq 0.8, 0.2 \leq x_2 \leq 0.8, 0.2 \leq x_3 \leq 0.8\}$$

Approximate solution $\bar{J}$ was found with 366 basis functions having been taken.
Graphics of $|\bar{J}^1|$ and for $|\bar{J}|$ in the cuts $x_3 = 0.4$ and $x_3 = 0.6$ are presented below.

Graphics for $|\bar{J}^1|$

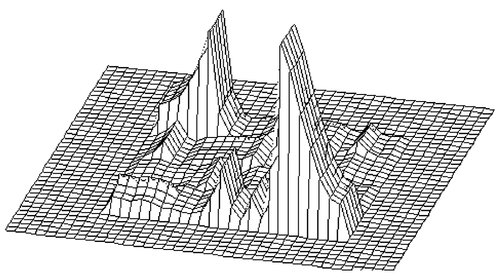


$\mathbf{x_3 = 0.6}$

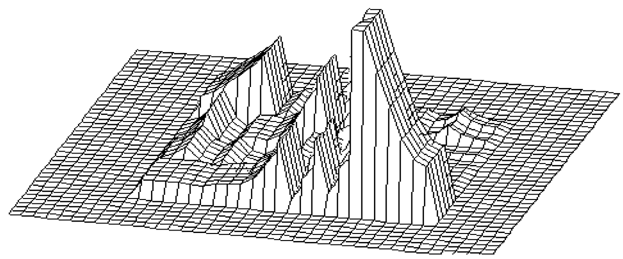


$\mathbf{x_3 = 0.6}$

Graphics for $|\bar{J}|$



$\mathbf{x_3 = 0.4}$



$\mathbf{x_3 = 0.6}$

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