HIERARCHICAL MULTI-LEVEL AND MULTI-GRID PRECONDITIONERS FOR ROBUST FEM-BASED ELECTROMAGNETIC MODELING

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ABSTRACT

Multi-grid and multi-level pre-conditioners have been shown to be effective in tackling some of the sources of ill-conditioning in standard finite element approximations of Maxwell's equations. The attributes of these two classes of preconditioning techniques are examined in this paper in the context of the various classes of boundary value problems that arise in antenna and microwave circuit analysis and design as well as the modeling of electromagnetic scattering and radiation problems. It is shown that with the proper choice of the preconditioning technique a very robust and fast converging FEM analysis can be performed for all the aforementioned classes of electromagnetic problems.

INTRODUCTION

The finite element method (FEM) is one of the most effective and versatile techniques for the modeling of complex microwave devices. Its application to practical engineering problems often results in large linear systems requiring iterative methods for their numerical solution. However, the convergence of iterative solvers tends to be unpredictable for electromagnetic wave problems, even when common pre-conditioners, such as incomplete LU factorization, are used to improve convergence. The reasons for the slow convergence of the iterative solver are by now well understood. They are associated with the dc modes contained in the null space of the curl operator, and the ill-conditioning of the FEM matrix resulting from the over-sampling of the low-frequency physical modes. Spurious dc modes can be canceled through the introduction of a spurious electric charge and the enforcement of the divergence free nature of the electric flux density explicitly in the weak statement of the electromagnetic problem. Use of the vector-scalar potential formulation for the development of the FEM approximation is most suitable for this purpose [1]. On the other hand, the difficulties associated with low-frequency physical modes can be tackled effectively by solving problems tentatively on coarser grids or in lower-order basis function spaces [2] More specifically, those modes that are over-sampled on the original FEM grid can be solved without loss of accuracy using an FEM system with much fewer degrees of freedom. Subsequently, through an interpolation process, the generated numerical solutions are projected back onto the original grid on which the higher-frequency modes are to be calculated accurately.

The demonstrated success of these remedies prompted their combination into a multi-grid vector-scalar potential finite element pre-conditioner that has been shown to exhibit outstanding convergence in conjunction with the analysis of three-dimensional electromagnetic problems [3]. There are two types of multi-grid techniques, geometric and hierarchical. The geometric multi-grid technique uses a set of nested grids obtained by dividing each tetrahedron in the coarsest grid into eight equal-volume sub-tetrahedra; hence, the geometric multi-grid technique functions as an h-adaptive finite element method. However, for those cases where the domain under study contains sub-domains where the electromagnetic field variation is sufficiently smooth, p-adaptive schemes can tackle numerical dispersion error more effectively than h-adaptive ones [4]. Therefore, for such cases a hierarchical multilevel vector-scalar potential finite element pre-conditioner is more suitable. Such a pre-conditioner uses only one grid and a set of hierarchical basis function spaces, i.e. H0(curl) and H1(curl), for the FEM approximation and numerical solution of the electromagnetic boundary value problem.

The attributes of these two classes of preconditioning techniques for FEM-based electromagnetic field solvers are examined in this paper in the context of the various classes of boundary value problems that arise in antenna and microwave circuit analysis and design as well as the modeling of electromagnetic scattering and radiation problems. It is shown that with the proper choice of the preconditioning technique a very robust and fast converging FEM analysis can be performed for all the aforementioned classes of electromagnetic problems.

HIERARCHICAL MULTI-LEVEL/MULTI-GRID POTENTIAL PRECONDITIONER

The details of the potential formulation of the electromagnetic boundary value problem have been discussed in detail elsewhere [1]-[5] and will not be repeated here. The FEM matrix equation obtained from the vector-scalar (or *A*-*V*) potential formulation through a Galerkin's process may be cast in the form,

$$\begin{bmatrix} M_{AA} & M_{AV} \\ M_{VA} & M_{VV} \end{bmatrix} \begin{bmatrix} x_A \\ x_V \end{bmatrix} = \begin{bmatrix} f_A \\ f_V \end{bmatrix} \tag{1}$$

where the vectors x_A and x_V contain, respectively, the expansion coefficients in the FEM approximation of $\vec{A}$ and V. Of major importance in the efficient implementation of the *A*-*V* formulation is the fact that the formulation is equivalent to the familiar electric field formulation (or *E* formulation) based on the FEM approximation of the vector Helmholtz equation. The FEM matrix equation obtained in this case will be denoted as follows for the purposes of this paper,

$$M_{EE} x_E = f_E \tag{2}$$

Because of this equivalence, a linear transformation exists between the two formulations, which makes possible the solution of (1) to be performed at essentially the same computational cost with that of the standard *E* formulation. The details can be found in [6]-[8].

For the application of a multi-grid process, in the context of a tetrahedron-based FEM discretization, nested grids are utilized that are obtained by dividing each tetrahedron in the coarser grid into eight equal-volume sub-tetrahedra. Inter-grid operators I_h^{2h} and I_{2h}^{h} are constructed and used to map the residual and the correction terms during the multi-grid process between adjacent (in coarseness) grids [5]. The edge elements used in the nested multi-grid technique are the lowest-order expansion functions. Thus, as it is well known, they are not as efficient in reducing the numerical dispersion error as higher-order expansion functions. This is the reason why a multi-level method where approximation accuracy is enhanced through the introduction of higher-order expansion functions, is more appropriate for those cases where such refinement makes sense. For the implementation of a hierarchical multi-level process a single grid is used with two (or more) sets of hierarchical basis functions, i.e., $H^0(curl)$ and $H^1(curl)$. A methodology for the systematic construction of hierarchical basis functions on tetrahedral was presented in [8]. The numerical implementation of a hierarchical multi-level process requires the construction of inter-level operators I_i^{i-1} and I_{i-1}^{i}. Their construction is discussed in [6].

We are now in a position to describe a hybrid multi-level/multi-grid process that acts, in effect, as a pre-conditioner for the numerically stable iterative solution of the FEM matrix of the *E* formulation. The pseudo-code description of the algorithm appeared in [7] and is repeated here for convenience. During the iterative solution of (2), the residual equation $M_{EE} z_E = r_E$ is involved. The hybrid pre-conditioner proceeds as described in Fig. 1. It is noted that the nested multi-grid pre-conditioning $\mathrm{MG}(z_E, r_E, i, j)$ is embedded in the solution of the $H^0(curl)$ matrix equation of the hierarchical multilevel pre-conditioner. Use of the *A*-*V* formulation is being made during the smoothing process both in the nested and the hierarchical portions of the algorithm.

APPLICATIONS AND CONCLUDING REMARKS

To provide an example from the application of the proposed pre-conditioners to the FEM modeling of waveguide structures the rectangular waveguide band-pass filter of [9] was analyzed. The geometry of the filter is depicted in the insert of Fig. 2. The cross-sectional dimensions of the guide are 22.86 mm x 10.16 mm. The distance between the top resonators is 19.63 mm, and their heights 16.54 mm. The height of the bottom resonator is 16.94 mm. The irises of the top resonators are 12.22 mm wide and 3.05 mm thick, while that of the bottom resonator is 11.63 mm wide and 3.05 mm thick. For the FEM analysis of this structure a hybrid 2-level multi-grid and 2-level multi-level pre-conditioner was implemented. The number of unknowns in $H^1(curl)$ is 152,062 while in $H^0(curl)$ is 26,375. However, because of the use of a multi-grid process within the multi-level process, the number of unknowns on the coarsest grid for the $H^0(curl)$ approximation, which dictates the dimension of the matrix that needs to be factored, is only 2,784. Figure 2 compares the calculated scattering parameters to measured data. The agreement is excellent across the entire frequency band. Figure 3 provides a pictorial description of the rapid convergence facilitated by the proposed pre-conditioning process. All simulations were run on a 600 MHz Pentium III processor. The memory requirement for the simulation was 53 Mbytes while the average CPU time per frequency point was 150 s.

1. $z_E \leftarrow 0$
2. $if\ i == 0$ // Nested Multi-grid
   - 2a. $if\ j == 0$
     - then solve $M_{EE}^{h} z_E = r_E$ // coarsest grid
   - 2b. *else*
     - 2b.1 Smooth (z_E, r_E) for v_1 times
     - 2b.2 $r_E^{2h} \leftarrow I_h^{2h}(r_E - M_{EE}^{h} z_E)$ and $z_E \leftarrow 0$
     - 2b.3 $\mathrm{MG}(z_E^{2h}, r_E^{2h}, 0, j-1)$
     - 2b.4 $z_E \leftarrow z_E + I_{2h}^{h} z_E^{2h}$
     - 2b.5 Smooth (z_E, r_E) for v_2 times
3. *else* //Hierarchical Multi-level
   - 3a. Smooth (z_E, r_E) for v_1 times
   - 3b. $r_E^{i-1} \leftarrow I_i^{i-1}(r_E - M_{EE}^{i} z_E)$ and $z_E^{i-1} \leftarrow 0$
   - 3c. $\mathrm{MG}(z_E^{i-1}, r_E^{i-1}, i-1, j)$
   - 3d. $z_E \leftarrow z_E + I_{i-1}^{i} z_E^{i-1}$
   - 3e. Smooth (z_E, r_E) for v_2 times

Figure 1. Pseudo-code for the hybrid multi-level/multi-grid pre-conditioner.

The hybridization of the multi-grid with the multi-level process is expected to be most suitable for those applications where the structure under modeling exhibits significant electrical size over which numerical dispersion must be contained. It is important to appreciate the fact that the application of the multi-grid process may result in rather coarse grids on which wavelength resolution could be as low as 3 points per wavelength. Our experience has been that unless dispersion error is contained at the coarsest level, the convergence rate of the iterative process is penalized. The implementation of the multi-level process in conjunction with helps address this problem. Finally, it is pointed out that for structures were fine geometric features control grid size the application of a multi-grid process may not be feasible. For such cases, the multi-level process without multi-grid may be utilized.

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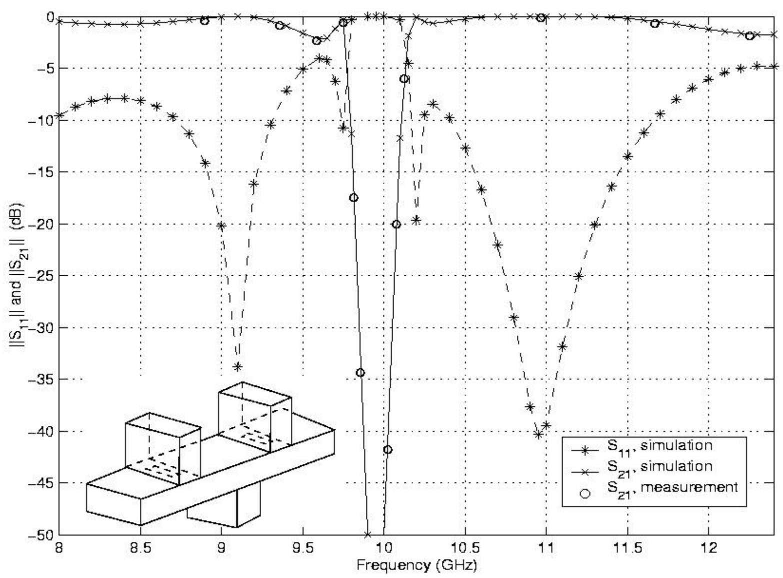

Figure 2. Magnitude of scattering parameters of the band-pass filter shown. Measured data obtained from [9].

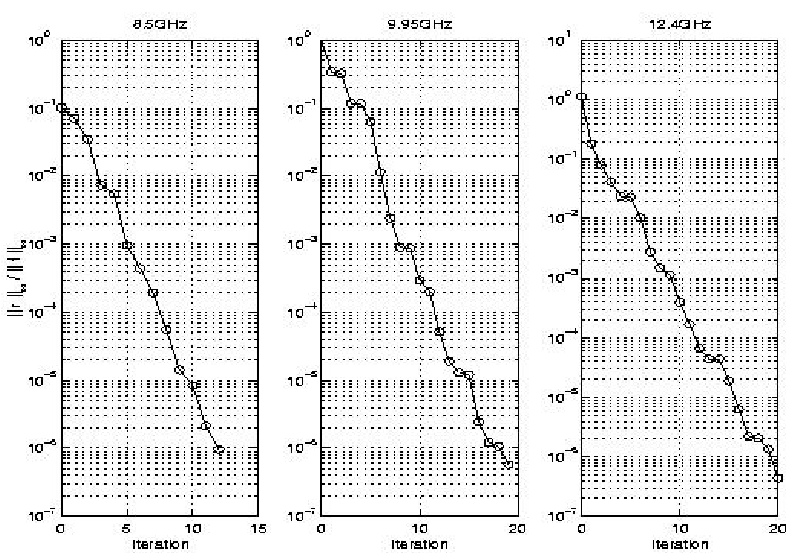

Figure 3. Convergence rate of the hybrid 2-level multilevel/2-level multigrid algorithm used for FEM pre-conditioning in the modeling of the band-pass filter.