# AN ALGEBRAIC APPROACH TO EIGENPROBLEMS AND ITS APPLICATION TO DOA ESTIMATION

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### ABSTRACT

This paper discusses the eigenproblems of correlation matrices in direction-of-arrival (DOA) estimation algorithms, especially for the case that the number of arriving waves is a few. The eigenvalues and eigenvectors can be obtained in a very short time (only 1.74% of the time by the conventional method) by regarding the eigenproblem as solving a fourth-order algebraic polynomial. It is also confirmed that the proposed algebraic approach does not make the accuracy worse when it is implemented by finite word-length processors like digital signal processor (DSP) or field programmable gate array (FPGA).

## INTRODUCTION

Mobile communication systems are intensively developing toward the next generation technology. To distinguish the target wave with the interference waves, DOA estimation is very significant for digital beam forming (DBF) array antenna system. Many DOA estimation algorithms have been already proposed [1]–[3], nowadays MUSIC [1] and ESPRIT [2] are two representative algorithms that can estimate DOAs accurately. The problem of the DOA estimation algorithms with correlation matrices (like MUSIC and ESPRIT) may be the computational complexity. One of the time-consuming processes in DOA estimation is computing eigenvalues/eigenvectors of correlation matrices.

Recalling that the derivation of eigenvalues is equivalent to solve the characteristic polynomial of the correlation matrix, the algebraic solvent can be applied if the order of the polynomial is four or less. This corresponds to the situation of DBF array antenna with four or less antenna elements (also valid when using a space averaging technique for every four or less elements). This paper presents a fast algebraic algorithm to derive eigenvalues/eigenvectors of correlation matrices, supposed to be used in the real-time DOA estimation for a small number of arriving waves [4]. The proposed algorithm employs the algebraic solvent of fourth-order characteristic polynomial to derive eigenvalues of correlation matrices instead of the numerical QR decomposition algorithm. The algebraic approach tends to make the accuracy worse when it is implemented by a digital device due to the quantization, however, the proposed approach guarantees that the quantization error does not affect to the estimated DOA when it is implemented by finite word-length processors like DSP or FPGA.

## PRELIMINARIES

Figure 1 illustrates the configuration of an (linear) array antenna system. The computation procedure of MUSIC method can be roughly summarized as the following three steps, for instance to see how the eigenvalues and eigenvectors are used in DOA estimation.

**[STEP 1]** The correlation matrix $\boldsymbol{R}_{xx}$ of the input vector $\boldsymbol{X}$ is obtained by

$$\boldsymbol{R}_{xx} = E\left[\boldsymbol{X}(t)\boldsymbol{X}^H(t)\right] = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^H + \sigma^2\boldsymbol{I},$$

where $\boldsymbol{X}$, $\boldsymbol{A}$ and $\boldsymbol{S}$ are so-called the signal, the direction and the signal correlation matrices, respectively.

**[STEP 2]** Derive the eigenvalues $\lambda_i$ and the corresponding eigenvectors $\boldsymbol{y}_i$ of the correlation matrix $\boldsymbol{R}_{xx}$.

**[STEP 3]** Using the direction matrix $\boldsymbol{A}$ and the eigenvectors $\boldsymbol{y}_i$, calculate the MUSIC spectrum $\boldsymbol{P}(\theta)$.

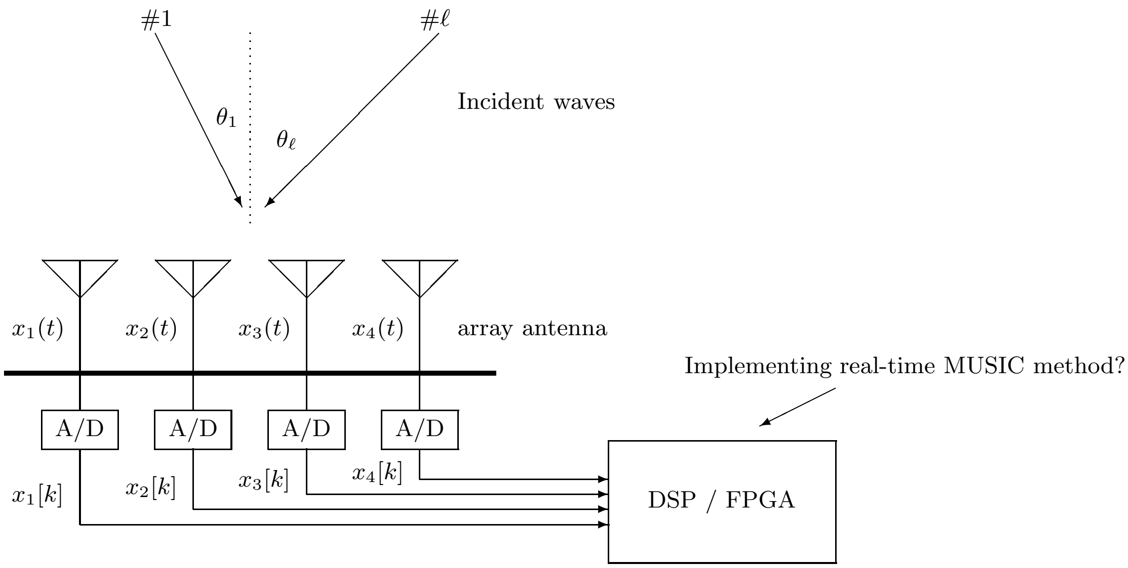



Figure 1: Configuration of array antenna system (with four elements)

In MUSIC method, the time-consuming process is usually either STEP 2 or STEP 3, and some fast algorithms have been already proposed for STEP 3. Representative one would be Root-MUSIC method [3] which is an algebraic algorithm without the direction search. Similarly, in the next section, we aim at developing an algebraic approach for STEP 2 to make this process faster.

## PROPOSED APPROACH

In this section, a fast algebraic algorithm is investigated for the DOA estimation by array antenna with four elements. The eigenvalues $\lambda_i$ of the matrix $\boldsymbol{R}_{xx}$ can be derived by solving the characteristic polynomial

$$\det(\boldsymbol{R}_{xx} - \lambda \boldsymbol{I}) = 0. \tag{1}$$

If the size of the matrix $\boldsymbol{R}_{xx}$ is $n \times n$, the characteristic polynomial becomes $n$-th order polynomial of $\lambda$, and it has $n$ solutions. Here, we adopt the well-known mathematical technique that the polynomials of up to fourth-order can be solved by an algebraic procedure. For the DOA estimation problems, we can establish an algebraic approach to derive eigenvalues of correlation matrices in case of four antenna elements, also when using a space averaging technique for every four elements. Now we investigate the algebraic approach.

### Deriving Eigenvalues

In case of four antenna elements, (1) can be reduced into the following fourth-order polynomial equation of $\lambda$:

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0. \tag{2}$$

Since the matrix $\boldsymbol{R}_{xx}$ is a non-negative Hermitian matrix with a full rank 4 [5], it has four real eigenvalues different to each other. By simply solving th equation (2), it's simple and well-known, the eigenvalues of the matrix $\boldsymbol{R}_{xx}$ can be derived as

$$\lambda_i = \sqrt{t} \pm \sqrt{h_1 \pm 2h_2} - \frac{a}{4}$$

where $h_1$, $h_2$ and $t$ are the functions of $a, b, c$ and $d$.

### Deriving Eigenvectors

Define the matrix $\boldsymbol{D}$ by

$$\boldsymbol{D} = [d_{ij}] := \boldsymbol{R}_{xx} - \lambda_i \boldsymbol{I} \in \boldsymbol{C}^{4\times 4},$$

then the eigenvector $\boldsymbol{y}_i = [y_{i1}, y_{i2}, y_{i3}, y_{i4}]^T$ of the matrix $\boldsymbol{D}$ corresponding to the $i$-th eigenvalue can be obtained by solving the following matrix equation:

$$(\boldsymbol{R}_{xx} - \lambda_i \boldsymbol{I})\boldsymbol{y}_i = \boldsymbol{D}\boldsymbol{y}_i = \boldsymbol{0}. \tag{3}$$

Equation (3) is generally solved by Gauss elimination method, however it is redundant since the matrix $\boldsymbol{D}$ is singular in this case. Hence, we employ a faster algorithm using an inverse matrix of the sub-matrix of $\boldsymbol{D}$. Since $\boldsymbol{D}$ is a singular matrix, the row-vectors $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3$ and $\boldsymbol{d}_4$ of the matrix $\boldsymbol{D}$ are linearly dependent. On the other hand, those four vectors include three linearly independent vectors. Suppose that $y_{i4} = 1$ and the vectors $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3$ are linearly independent. Based on this relation, the eigenvectors $\boldsymbol{y}_i$ can be calculated. (Indeed it's faster than Gaussian method, however space does not prove to show the results.)

## SIMULATION

The proposed fourth-order polynomial approach is evaluated in comparison with the general QR decomposition. Table 1 shows the computation times of the two methods, polynomial and QR decomposition. From Table 1, the polynomial approach requires only 1.74% of the computation time that required in the QR decomposition. From this fact, the proposed polynomial approach is very effective to shorten the computation time to solve eigenproblems. Indeed the computation time of this process can almost be ignored. Also note that we tested thousands of example input vectors, and the result in the Table 1 is the average of those trials.

Table 1: Comparison of the required computation time

| | QR decomp. | Proposed |
|---|---|---|
| computation time<br>(ratio) | 1.54msec<br>(1.000) | 24.2$\mu$sec<br>(0.0174) |
| required memories | 560KB | 572KB |

(CPU: Intel Celeron 433MHz, Memory:256MB)

## DISCUSSION

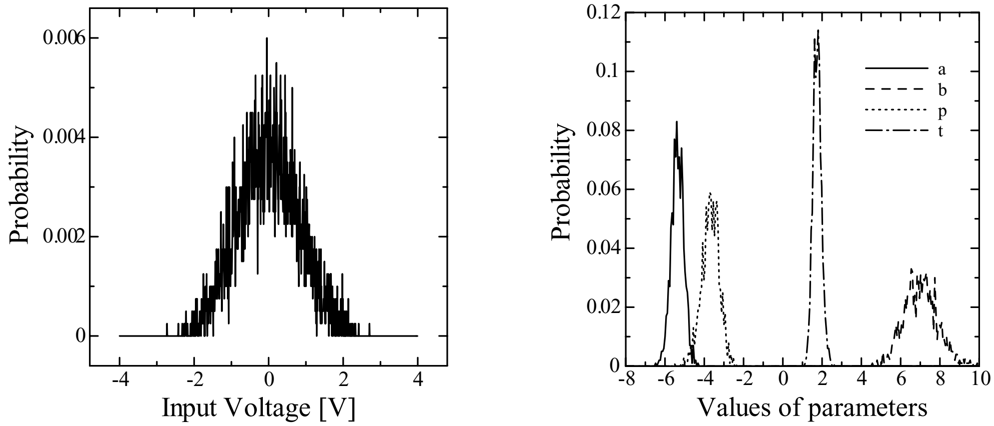
Assuming to implement the proposed algorithm by finite word-length (digital) devices, we confirm that the quantization noise caused by DSP/FPGA implementation does not make the DOA estimation accuracy worse.

### Effect of the Quantization Noise in A/D Converter

First, the effect of the quantization noise in A/D converter is studied. The A/D converter and its quantization noise are imperative in digital implementation. Here we assume that some signals with power one are coming to the array antenna. In this case, the distribution of the input voltage becomes as illustrated in Fig. 2(a). From Fig. 2(a), we see that the input voltage usually vibrates within $-2$ to 2 volts, and we can adjust the level of the A/D converter not to make overflow. Actually, it does not matter if the numbers more than 2 is rounded to 2. Figure 3(a) shows the effect of quantization noise in A/D converter to the estimated DOAs. As seen in Fig.3(a), eight quantization bit length in A/D converter is enough for the accurate DOA estimation. Although the noise level in Fig.3(a) becomes slightly larger, it does not affect to the accuracy of DOA estimation.

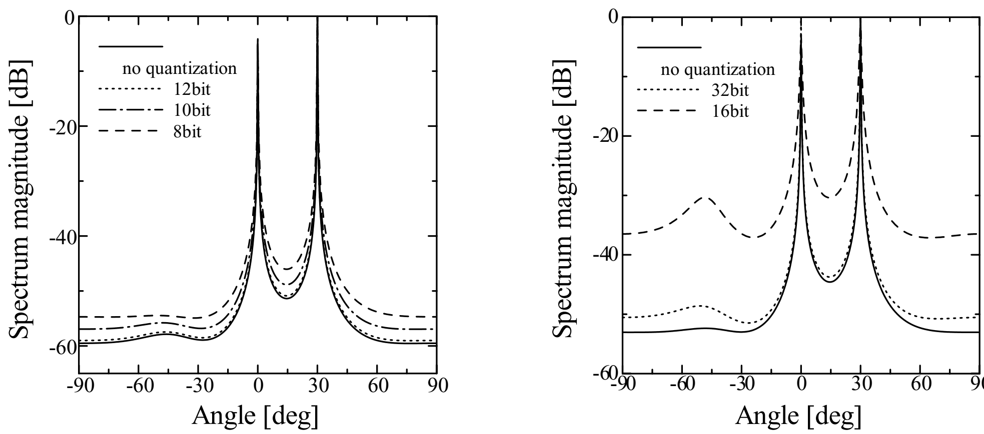
### Effect of the Quantization Noise in DSP

Next, we study the effect of the quantization noise in DSP. In the proposed algebraic approach, we should confirm the range of parameters in the middle of digital processing, not to make overflow. Figure 2(b) shows the example distribution of intermediate parameters. From Fig.2(b), the parameters vibrates within about 10-times of the range of the input voltage. Generally the operations in the fixed-point DSP are performed with 16-bits (single precision) or 32-bits (double precision). Considering the range of parameters, 16-bits operation would be enough for the accurate DOA estimation. Figure 3(b) shows the effect of the quantization noise in DSP. The quantization bit length in A/D converter is set to be eight. From Fig.3(b), the fixed-point DSP operation does not make the accuracy of the DOA estimation worse.




(a) (b)

Figure 2: Distribution of Parameters: (a) that of Input voltage, (b) that of example intermediate parameters




(a) (b)

Figure 3: Effects of quantization errors in DOA estimation: (a) that in A/D converter, (b) that in DSP

# CONCLUDING REMARKS

This paper focused on the eigenproblems for that the number of the arriving wave was a few, say, the order of the correlation matrix was four or less. Regarding the eigenproblem as solving the fourth-order algebraic polynomial, the eigenvalues and eigenvectors could be obtained in a very short time (only 1.74% of that by the conventional method). Moreover, we confirmed that the proposed algebraic approach does not make the accuracy worse when it was implemented by finite word-length processors.

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