APPLICATION OF INVERSE SCATTERING METHODS TO THE MATERIAL CHARACTERIZATION OF SUBSURFACE OBJECTS[1]

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ABSTRACT

The problem of material characterization for an inhomogeneous medium with arbitrary dispersion is posed as an inverse scattering problem. In this respect, two sub-problems are identified and considered: multi-frequency reconstruction of the spatial variation for a given law of dispersion, and reconstruction of the dispersion law for a given spatial variation. Possibility of significant non-physical dispersion at low frequencies is predicted and demonstrated on experimental data.

INTRODUCTION

Nowadays, a range of devices is used for the characterization of inhomogeneous media using the electromagnetic field. To name a few: microwave medical scanners, geophysical resistometers, scanning near-filed microscopes, etc. The main purpose of these devices is to visualize the structure of the subsurface and classify the materials present. Depending on the frequency-band of the applied electromagnetic field different mathematical models of the scattering process are used, ranging from the Laplace equation for the static field, trough the classical macroscopic Maxwell's equations, to the microscopic equations of quantum physics. We consider here those frequencies, for which the macroscopic Maxwell's equations are applicable. This model is suitable, for example, in the case of the Ground Penetrating Radar (GPR) operating at 10^2 – 5×10^9 Hz. The magnitude of the generated field is small enough to allow for a linear model of the field-medium interaction. In the frequency domain any isotropic medium is adequately described by the constitutive parameters: scalar spatially varying complex-valued functions of the dielectric permittivity $\varepsilon(\mathbf{r}, \omega)$, magnetic permeability $\mu(\mathbf{r}, \omega)$ and conductivity $\sigma(\mathbf{r}, \omega)$. As indicated, these functions depend on the applied frequency ω.

As far as the target classification is concerned, these functions contain all the information (inherent to the target, not to the scattered field!), which can possibly be obtained from the GPR measurements. On one hand, this means that if at a given frequency-band two different objects are characterized by the same triple $\{\varepsilon(\mathbf{r}, \omega),\ \mu(\mathbf{r}, \omega),\ \sigma(\mathbf{r}, \omega)\}$, then these objects are indistinguishable for the GPR. On the other hand, this spares us a lot of effort, since there is nothing more in the data.

The problem of retrieving the triple $\{\varepsilon(\mathbf{r}, \omega),\ \mu(\mathbf{r}, \omega),\ \sigma(\mathbf{r}, \omega)\}$ from the scattered field data is called the inverse scattering problem. It is a well established fact that this problem is ill-posed. As a consequence, in realistic conditions exact reconstruction of the mentioned triple can never be achieved. At most, an approximate triple $\{\tilde{\varepsilon}(\mathbf{r}, \omega),\ \tilde{\mu}(\mathbf{r}, \omega),\ \tilde{\sigma}(\mathbf{r}, \omega)\}$ will be obtained. This loss of information also degrades the classification capabilities of GPR, since even objects, which are distinct in terms of their exact triples, may become indistinguishable in terms of the reconstructed approximate triples.

Most of the effort in inverse scattering is devoted to the reconstruction of the spatial variation of the constitutive parameters. Indeed, to have the exact spatial variation of $\{\varepsilon(\mathbf{r}, \omega),\ \mu(\mathbf{r}, \omega),\ \sigma(\mathbf{r}, \omega)\}$ for some buried object is almost equivalent to seeing this object with your own eyes. This is, however, not achievable, since the approximate reconstruction $\{\tilde{\varepsilon}(\mathbf{r}, \omega),\ \tilde{\mu}(\mathbf{r}, \omega),\ \tilde{\sigma}(\mathbf{r}, \omega)\}$ contains significant spatial blurring. The de-blurring algorithms sometimes allow to obtain sharper images, however, often, at a price of error in the magnitude of the reconstructed functions.

In this paper we concentrate on obtaining the actual magnitude of the constitutive parameters. In addition, we would like to obtain information about the frequency dependence of these parameters. It appears, that this dependence, also known as dispersion, may be driven by both physical and non-physical factors. We shall base our discussion on the two-dimensional inverse scattering problems related to the GPR configuration, where the constitutive parameters reduce in the TM-case to $\{\varepsilon(\mathbf{r}, \omega),\ \sigma(\mathbf{r}, \omega)\}$.

THEORY

In this section we start with the general statement of the inverse scattering problem with multi-frequency data, followed

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by the analysis of physical and non-physical dispersion of the electromagnetic constitutive parameters.

Multi-Frequency Inversion

Consider a fixed-offset (bi-static) GPR. The time-domain scattered field data-set $f^{\rm sc}(\mathbf{r}^{\rm r},t)$ is collected over an equispaced M-point grid along a segment of straight line S comprising the so-called B-scan. The Fourier image $u^{\rm sc}(\mathbf{r}^{\rm r},\omega)$ of the time-domain data is the frequency-domain scattered field. At each frequency ω this field satisfies the well-known set of integral equations, which we write as follows:

$$\left\{\begin{array}{l} R(u\chi) = u^{\rm sc} \\ u - G(u\chi) = u^{\rm in} \end{array}\right. \quad \rightarrow \quad \left\{\begin{array}{l} R_k\,{\rm d}(\chi_k)\,u_k = u_k^{\rm sc} \\ \left[I - G_k\,{\rm d}(\chi_k)\right]\,u_k = u_k^{\rm in} \end{array}\right. . \tag{1}$$

On the left we have the continuous form of these equations in operator notation, where $u(\mathbf{r},\omega)$ is the total and $u^{\rm in}(\mathbf{r},\omega)$ is the incident field inside D – a given sub-domain of the subsurface, where the contrast-function $\chi(\mathbf{r},\omega)$ is different from zero. Since we deal with an arbitrary two-dimensional spatial configuration, which can be treated only numerically, it makes sense to work with the discrete analogue of the problem. Therefore, on the right, equations are given in their discrete form, where ω takes values from the set $\{\omega_k,\ k=1,2,\dots K\}$, and the frequency-dependence is shown in the form of a subscript. Further, $R_k \in \mathbb{C}^{M\times N}$, $G_k \in \mathbb{C}^{N\times N}$, $\{u_k,\ u_k^{\rm in},\ \chi_k\} \in \mathbb{C}^N$, and $u_k^{\rm sc} \in \mathbb{C}^M$. As one can see, after the discretization on a N-point two-dimensional grid the resulting matrix equations assume essentially the same form as their continuous counterparts. However, the product $u(\mathbf{r},\omega)\chi(\mathbf{r},\omega)$ of two functions on the left becomes now the Hadamard (entrywise) product $(u_k \circ \chi_k)$ of two vectors of the same size on the right. We employ the following relation: $u_k \circ \chi_k = {\rm d}(u_k)\,\chi_k = {\rm d}(\chi_k)\,u_k$, where ${\rm d}(x) = {\rm diag}(x)$ is a diagonal matrix with vector x on its diagonal. Substituting the formal solution of the lower equation with respect to u_k into the upper one we obtain the discrete nonlinear equation of the inverse scattering problem

$$R_k\,{\rm d}(\chi_k)\,[I - G_k\,{\rm d}(\chi_k)]^{-1} u_k^{\rm in} = u_k^{\rm sc}, \tag{2}$$

with χ_k being the fundamental unknown. This problem can be approximately solved using different numerical techniques. In particular, we have studied the Contrast-Source approach, and the method of effective inversion. If the contrast is low, then the Born approximation can be applied

$$R_k\,{\rm d}(u_k^{\rm in})\,\chi_k \approx u_k^{\rm sc}. \tag{3}$$

With this linear ill-posed problem different regularization techniques can be used. Note, however, that if $\chi(\mathbf{r},\omega)$ has arbitrary (unspecified) frequency-dependence, then we deal with K different problems and multi-frequency data do not help to solve any of them.

Physical Dispersion

In practice, we usually presume some particular type of frequency-dependence, most often, the static model of a conducting dielectric, i.e.

$$\chi(\mathbf{r},\omega) = \varepsilon(\mathbf{r}) - \varepsilon_{\rm b} + \frac{i}{\omega}[\sigma(\mathbf{r}) - \sigma_{\rm b}] = \phi^{\varepsilon}(\mathbf{r}) + s(\omega)\phi^{\sigma}(\mathbf{r}), \tag{4}$$

where $\varepsilon_{\rm b}$, and $\sigma_{\rm b}$ denote the constant permittivity and conductivity of the background medium. Using (4) in (3) we obtain

$$\begin{pmatrix} R_1\,{\rm d}(u_1^{\rm in}) & 0 & . & . & 0 \\ 0 & R_2\,{\rm d}(u_2^{\rm in}) & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & R_K\,{\rm d}(u_K^{\rm in}) \end{pmatrix} \begin{pmatrix} I & s_1 I \\ I & s_2 I \\ . & . \\ . & . \\ I & s_K I \end{pmatrix} \begin{pmatrix} \phi^{\varepsilon} \\ \phi^{\sigma} \end{pmatrix} \approx \begin{pmatrix} u_1^{\rm sc} \\ u_2^{\rm sc} \\ . \\ . \\ u_K^{\rm sc} \end{pmatrix}, \tag{5}$$

where $s_k = s(\omega_k)$ are complex constants, I is $N\times N$ identity matrix, and $\phi^{\varepsilon/\sigma} \in \mathbb{R}^N$. Carrying out multiplication of the matrices in the left-hand side and resorting to the normal form of this equation, which describes its least squares solution, we get

$$\begin{pmatrix} \sum\limits_{k=1}^{K} {\rm d}(u_k^{\rm in})^H\,R_k^H R_k\,{\rm d}(u_k^{\rm in}) & \sum\limits_{k=1}^{K} s_k\,{\rm d}(u_k^{\rm in})^H\,R_k^H R_k\,{\rm d}(u_k^{\rm in}) \\ \sum\limits_{k=1}^{K} \overline{s}_k\,{\rm d}(u_k^{\rm in})^H\,R_k^H R_k\,{\rm d}(u_k^{\rm in}) & \sum\limits_{k=1}^{K} |s_k|^2\,{\rm d}(u_k^{\rm in})^H\,R_k^H R_k\,{\rm d}(u_k^{\rm in}) \end{pmatrix} \begin{pmatrix} \phi^{\varepsilon} \\ \phi^{\sigma} \end{pmatrix} \approx \begin{pmatrix} \sum\limits_{k=1}^{K} {\rm d}(u_k^{\rm in})^H\,R_k^H u_k^{\rm sc} \\ \sum\limits_{k=1}^{K} \overline{s}_k\,{\rm d}(u_k^{\rm in})^H\,R_k^H u_k^{\rm sc} \end{pmatrix} \tag{6}$$

This problem is not equivalent to K independent problems (3), and adding data at new frequencies may improve the reconstruction of ϕ^{ε} and ϕ^{σ}, as was observed on synthetic models. Hence, in order to use the multi-frequency data efficiently one needs to know the explicit form of the frequency-dependence for the materials at hand. This is, however, not enough. As we explain below, certain structural assumptions about the object are required as well.

A general dispersion law for an isotropic conducting dielectric, which is still suitable for the classical macroscopic treatment, has the following form (see e.g. G. R. Fowles, *Introduction to Modern Optics*, 2nd ed., Dover: New York, pp. 152–163, 1989):

$$\varepsilon(\mathbf{r},\omega) = \varepsilon_0 + \frac{N^{\varepsilon}(\mathbf{r})\, e^2}{m}\sum_j \frac{\nu_j}{\omega_j^2(\mathbf{r}) - \omega^2 - i\gamma_j(\mathbf{r})\,\omega}, \qquad \sigma(\mathbf{r},\omega) = \frac{N^{\sigma}(\mathbf{r})\, e^2/m}{1 - i\tau^{\sigma}(\mathbf{r})\,\omega}, \tag{7}$$

where $e^2/m \approx 2.818\times 10^{-8}\,\mathrm{C}^2\,\mathrm{kg}^{-1}$, $N^{\varepsilon/\sigma}(\mathbf{r})$ are concentrations of the bound and free electrons per unit volume, $\omega_j(\mathbf{r})$ – resonance frequencies of the bound electrons, $\gamma_j(\mathbf{r})$ – damping factors, ν_j – oscillator strengths, and $\tau^{\sigma}(\mathbf{r})$ – relaxation time. As indicated in the formula, the spatial variation of $\varepsilon(\mathbf{r},\omega)$ and $\sigma(\mathbf{r},\omega)$ may be caused by different factors. The resonance frequencies become important starting from the infrared frequencies. In the absence of restoring force only the damping factor and the relaxation time are important we take the limit $\omega_j \to 0$, and skipping the j-dependence, we obtain

$$\varepsilon(\mathbf{r},\omega) = \varepsilon_0 - \frac{N(\mathbf{r})\, e^2/m}{\omega^2 + i\gamma(\mathbf{r})\,\omega}, \tag{8}$$

Clearly, there is no distinction between the complex conductivity and complex permittivity in this situation, since all electrons move freely now. By definition, the incident field $u^{\mathrm{in}}(\mathbf{r},\omega)$ is the one excited by external sources in a homogeneous background with $\varepsilon_{\mathrm{b}}(\omega) = \varepsilon_0 - (N_{\mathrm{b}}\, e^2/m)/(\omega^2 + i\gamma_{\mathrm{b}}\omega)$. Explicitly, the complex-valued electromagnetic contrast is given by

$$\chi(\mathbf{r},\omega) = \frac{e^2}{m}\left[\frac{N_{\mathrm{b}}}{\omega^2+\gamma_{\mathrm{b}}^2} - \frac{N(\mathbf{r})}{\omega^2+\gamma^2(\mathbf{r})}\right] + i\frac{e^2}{m\,\omega}\left[\frac{N(\mathbf{r})\,\gamma(\mathbf{r})}{\omega^2+\gamma^2(\mathbf{r})} - \frac{N_{\mathrm{b}}\gamma_{\mathrm{b}}}{\omega^2+\gamma_{\mathrm{b}}^2}\right], \tag{9}$$

with $N(\mathbf{r}) = N_{\mathrm{b}}$ and $\gamma(\mathbf{r}) = \gamma_{\mathrm{b}}$ for $\mathbf{r} \notin D$. In general, for the efficient multi-frequency inversion (6), we have to be able to represent $\chi(\mathbf{r},\omega)$ in the form

$$\chi(\mathbf{r},\omega) = F(\omega)\,\phi(\mathbf{r}), \tag{10}$$

where $F(\omega)$ is an operator (depending on ω), which is linear with respect to the spatial function $\phi(\mathbf{r})$. Although, in general, the contrast given in (9) does not allow for this representation, there exist situations where something can be done. Suppose, an object consists of materials with the same damping factor (relaxation time) as the one of the background medium $\gamma(\mathbf{r}) = \gamma_{\mathrm{b}}$. Then, we obtain

$$\chi(\mathbf{r},\omega) = \frac{e^2/m}{\omega^2 + i\gamma_{\mathrm{b}}\,\omega}\left[N_{\mathrm{b}} - N(\mathbf{r})\right] = s^{\mathrm{b}}(\omega)\,\phi(\mathbf{r}). \tag{11}$$

So, that the discrete multi-frequency inverse problem with respect to the difference in concentrations once again acquires a simple normal form

$$\left[\sum_{k=1}^{K} |s_k^{\mathrm{b}}|^2\, \mathrm{d}(u_k^{\mathrm{in}})^H\, R_k^H R_k\, \mathrm{d}(u_k^{\mathrm{in}})\right]\phi = \sum_{k=1}^{K} \overline{s_k^{\mathrm{b}}}\, \mathrm{d}(u_k^{\mathrm{in}})^H\, R_k^H u_k^{\mathrm{sc}}. \tag{12}$$

Another simple case is the homogeneous object with a given spatial support. In this case $N(\mathbf{r}) = N_{\mathrm{s}}$ and $\gamma(\mathbf{r}) = \gamma_{\mathrm{s}}$, so that $\chi(\mathbf{r},\omega) = \chi(\omega)$, and each of the problems (3) reduces to

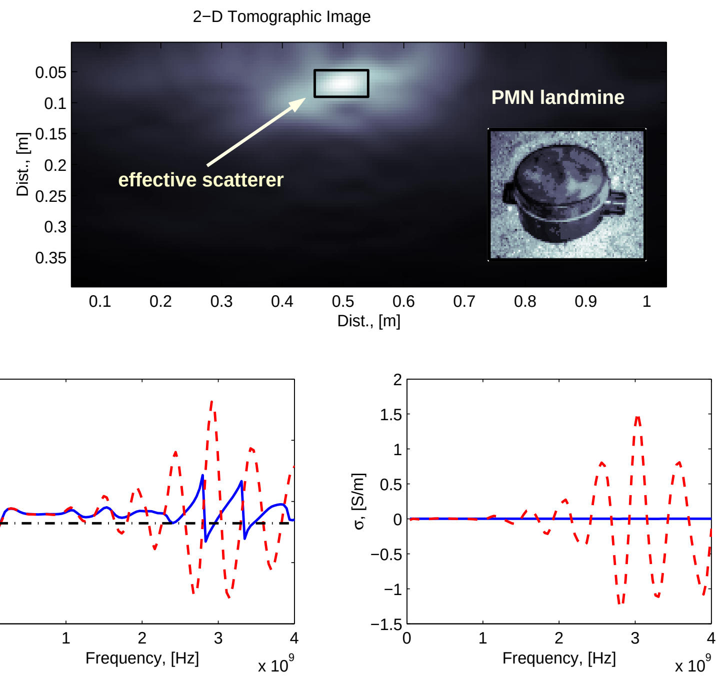
$$\|R_k\, u_k^{\mathrm{in}}\|^2 \chi_k = (R_k\, u_k^{\mathrm{in}})^H u_k^{\mathrm{sc}}, \quad k = 1, 2, \ldots K. \tag{13}$$

At the moment, we do not know other cases, where there would be any significant simplification of the problems (2)–(3) with contrast given by (9). Therefore, in general, the multi-frequency inversion problem has to be treated as a set of independent single-frequency inverse problems.

Non-Physical Dispersion

Consider the last example (13) of the previous sub-section, which represents the inverse problem with respect to the frequency-dependence of the contrast. For $\|R_k\, u_k^{\mathrm{in}}\|^2 \neq 0$ we can write the least squares solution as

$$\chi_k = \frac{(R_k\, u_k^{\mathrm{in}})^H u_k^{\mathrm{sc}}}{\|R_k\, u_k^{\mathrm{in}}\|^2}, \quad k = 1, 2, \ldots K. \tag{14}$$



Obviously, the Born approximation applied in (3) introduces an error in the scattering model. In addition, matrices R_k may be given with an error, for example, when the spatial support of the object is not known exactly. Suppose, the scattered field $u_k^{\rm sc}$ is exact, and introduce the discrepancy

$$\delta\chi_k = \tilde{\chi}_k - \chi_k = \left(\frac{\tilde{R}_k\, \tilde{u}_k}{\|\tilde{R}_k\, \tilde{u}_k\|^2} - \frac{R_k\, u_k}{\|R_k\, u_k\|^2} \right)^H u_k^{\rm sc}, \quad k = 1, 2, \ldots K \tag{15}$$

between the true contrast χ_k and its computed version $\tilde{\chi}_k$. Here, $\tilde{R}_k$ is the approximate matrix, obtained by discretization of the approximate operator, and $\tilde{u}_k$ is the approximate total field. To a certain extent, the two errors can be analyzed separately. First, presume that $\tilde{R}_k = R_k$, and express $\tilde{u}_k = u_k + \delta u_k$. In the second case, we presume $\tilde{u}_k = u_k$, but $\tilde{R}_k = R_k + \Delta R_k$. The corresponding discrepancies are

$$\delta\chi_k = \left(\frac{R_k\, u_k + R_k\, \delta u_k}{\|R_k\, u_k + R_k\, \delta u_k\|^2} - \frac{u_k}{\|R_k\, u_k\|^2} \right)^H u_k^{\rm sc} \quad \text{and} \quad \delta\chi_k = \left(\frac{R_k\, u_k + \Delta R_k\, u_k}{\|R_k\, u_k + \Delta R_k\, u_k\|^2} - \frac{R_k}{\|R_k\, u_k\|^2} \right)^H u_k^{\rm sc}. \tag{16}$$

In the first case, the reconstruction is exact if $\delta u_k \in \mathcal{N}(R_k)$, where $\mathcal{N}(R_k)$ denotes the null-space of matrix R_k. This means that we do not need to know the exact total field u_k, but only its spectral components, which lie outside the null-space of exact matrix R_k. Whereas, in the second case, the reconstruction is exact if $u_k \in \mathcal{N}(\Delta R_k)$, where $\mathcal{N}(\Delta R_k)$ is the null-space of distortion-matrix ΔR_k. If the aforementioned errors do not satisfy these criteria and depend on frequency, some additional dispersion in the reconstructed $\tilde{\chi}_k$ will be introduced. Since the causes of such dispersion are purely mathematical, we call it *non-physical* or *model-related* dispersion.

EXPERIMENT

Probably, the easiest way to demonstrate the phenomenon of non-physical dispersion is to consider the two-dimensional inversion of a three-dimensional data-set. In this case, the error in the matrix R_k is both large and frequency-dependent. Consider typical GPR data collected by the DeTeC research group of the Swiss Federal Institute of Technology (EPFL). The central frequency of their radar was approximately 1 GHz, while the objects were plastic antipersonnel landmines with sizes between 0.05 and 0.12 m. The explosive compound of landmines was substituted with RTV polymer material. The RTV's permittivity and conductivity are similar to those of TNT: $\varepsilon_{\rm RTV}/\varepsilon_0 \approx 2.88{-}3.13$, $\sigma_{\rm RTV} \approx 0.00001{-}0.00002$ S/m; $\varepsilon_{\rm TNT}/\varepsilon_0 \approx 2.89$, $\sigma_{\rm TNT} \approx 0.00002$ S/m, as measured at 300 MHz. Further details of the experimental setup can be found at the Internet address `http://diwww.epfl.ch/lami/detec/gprimages.html`. Application of the 2-D reflection travel-time tomography to one of the data-sets is shown in figure (top). At the bottom of this figure we show the frequency dependence of the reconstructed relative permittivity and conductivity obtained with linearized (dashed) and nonlinear (solid) inversion algorithms. Strong resonance-type dispersion of non-physical origin can be clearly seen.