UNIQUENESS IN INVERSE OBSTACLE SCATTERING FOR ELECTROMAGNETIC WAVES

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ABSTRACT

The inverse problem we consider in this survey is to determine the shape of an obstacle from the knowledge of the far field pattern for the scattering of time-harmonic electromagnetic waves. We will concentrate on uniqueness issues, i.e., we will investigate under what conditions an obstacle and its boundary condition can be identified from a knowledge of its far field patterns for incident plane waves. We will review recent uniqueness results and draw attention to open problems. Furthermore, we will illustrate the connection of the uniqueness proofs to some numerical methods for the approximate solution of the nonlinear and improperly posed inverse obstacle problem that have been developed recently, such as the sampling methods due to Colton and Kirsch and the point-source methods due to Potthast.

THE INVERSE OBSTACLE SCATTERING PROBLEM

The propagation of time-harmonic electromagnetic waves in a homogeneous isotropic medium in $\mathbb{R}^3$ is modelled through the reduced *Maxwell equations*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0. \tag{1}$$

Here E and H denote the space dependent parts of the electric field $\varepsilon^{-1/2}E(x)e^{-i\omega t}$ and the magnetic field $\mu^{-1/2}H(x)e^{-i\omega t}$ respectively, k is the positive wave number given by $k = \sqrt{\varepsilon\mu}\,\omega$ in terms of the frequency ω, the electric permittivity ε and the magnetic permeability μ. The scattering of time-harmonic electromagnetic waves by an impenetrable bounded obstacle D in $\mathbb{R}^3$ (with a connected boundary ∂D) leads to exterior boundary value problems for the Maxwell equations. The total wave E, H must satisfy the Maxwell equations in the exterior $\mathbb{R}^3 \setminus \bar{D}$ of the scatterer and is decomposed $E = E^i + E^s$, $H = H^i + H^s$ into the given incident wave E^i, H^i and the unknown scattered wave E^s, H^s which is required to satisfy the *Silver–Müller radiation condition*

$$\lim_{|x|\to\infty} (H^s(x) \times x - |x|\, E^s(x)) = 0 \tag{2}$$

uniformly with respect to all directions. On the boundary ∂D the total wave has to satisfy a boundary condition of the form

$$B(E, H) = 0 \quad \text{on } \partial D \tag{3}$$

with the operator B depending on the nature of the scatterer. For a *perfect conductor* we have $B(E, H) = \nu \times E$, where ν denotes the unit outward normal to the boundary ∂D, i.e., the total electric field has a vanishing tangential component

$$\nu \times E = 0 \quad \text{on } \partial D.$$

A more general boundary condition is the *impedance* or *Leontovich boundary condition*

$$\nu \times H - \lambda(\nu \times E) \times \nu = 0 \quad \text{on } \partial D$$

with a positive function λ, that is, $B(E, H) = \nu \times H - \lambda\,(\nu \times E) \times \nu$. Throughout this exposition we assume the boundary ∂D of the scatterer to be of class C^2. Then existence of a solution and well-posedness for the above exterior boundary value problems can be based on boundary integral equations (see [2]).

The radiation condition (2) ensures uniqueness of the solution to the exterior boundary value problems and leads to an asymptotic behavior of the form

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right\}, \quad H^s(x) = \frac{e^{ik|x|}}{|x|}\left\{H_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right\} \tag{4}$$

as $|x| \to \infty$, where the vector fields E_∞ and H_∞ , defined on the unit sphere Ω in $\mathbb{R}^3$, are known as the *electric far field pattern* and *magnetic far field pattern* of the scattered wave, respectively. They satisfy $H_\infty = \nu \times E_\infty$ and $\nu \cdot E_\infty = \nu \cdot H_\infty = 0$ with the unit outward normal ν on Ω. A vanishing electric far field pattern $E_\infty = 0$ (or a vanishing magnetic far field pattern $H_\infty = 0$) on the unit sphere Ω implies $E^s = H^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by Rellich's lemma (see Theorem 6.9 in [2]).
An important cases of incident fields are plane waves

$$E^i(x, d, p) = p\, e^{ik\, x\cdot d}, \quad H^i(x, d, p) = d \times p\, e^{ik\, x\cdot d}$$

with propagation direction $d \in \Omega$ and polarization $p \perp d$. The corresponding scattered waves and far field patterns we denote by $E^s(\cdot\,, d, p)$, $H^s(\cdot\,, d, p)$ and $E_\infty(\cdot\,, d, p)$, $H_\infty(\cdot\,, d, p)$, respectively, indicating the dependence on the direction d and polarization p of the incident field. The *inverse problem* we are considering is, given the electric far field pattern $E_\infty(\cdot\,, d, p)$ (or the magnetic far field pattern $H_\infty(\cdot\,, d, p)$) for one or several incident plane waves with incident directions d and polarizations d to determine the shape of the scatterer D. For a comprehensive study of this inverse scattering problem we refer to Colton and Kress [2]. Here, we will concentrate on uniqueness issues, i.e., we will investigate under what conditions an obstacle and its boundary condition can be identified from a knowledge of its far field patterns for incident plane waves.

UNIQUENESS

Since by Rellich's lemma the far field pattern uniquely determines the scattered wave and consequently the total wave in the exterior of the scatterer, the question of uniqueness for the inverse problem is equivalent to the question whether the total wave can satisfy the boundary condition (3) for two different domains D_1 and D_2. We immediately can exclude the case where the two scatterers are disjoint, i.e., $\overline{D_1} \cap \overline{D_2} = \emptyset$. In this situation, the scattered wave is defined in all of $\mathbb{R}^3$, since it is defined in the exterior of both D_1 and D_2. Consequently, the scattered wave E^s, H^s is an entire solution to the Maxwell equations satisfying the radiation condition and therefore it must be identically zero. However, then the total wave coincides with the incident field and therefore the incident plane wave itself must satisfy the boundary condition. Both for the perfect conductor and the impedance boundary condition this leads to $\nu \times p = 0$ on ∂D and this is impossible for a closed surface. Hence, nonuniqueness can occur only when $\overline{D_1} \cap \overline{D_2} \neq \emptyset$, and, presently, this case cannot yet be excluded on the knowledge of the far field pattern for scattering of one incident plane wave only. However, when we have overdetermined data in the sense that the far field pattern is known for all incident directions and polarizations we have the following uniqueness result based on ideas of Kirsch and Kress [8] (see Theorem 7.1 in [2]).

Theorem 1 *Assume that D_1 and D_2 are two scatterers with boundary conditions B_1 and B_2 such that the far field patterns coincide for all incident directions d, all polarizations $p \perp d$, and all observation directions $\hat{x}$. Then $D_1 = D_2$ and $B_1 = B_2$.*

Proof. Following Potthast [15] we simplify the approach of Kirsch and Kress through the use of a mixed reciprocity relation. For this, in addition to scattering of plane waves we also consider scattering of electric dipole fields. Here, the incident field

$$E^i_e(x, y, p) = \frac{i}{k}\, \operatorname{curl}_x \operatorname{curl}_x p\Phi(x, y), \quad H^i_e(x, y, p) = \operatorname{curl}_x p\Phi(x, y)$$

due to an electric dipole with polarization p located at $y \in \mathbb{R}^3$ is given in terms of the fundamental solution to the Helmholtz equation

$$\Phi(x, y) := \frac{1}{4\pi}\, \frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y.$$

We denote the corresponding scattered waves by $E^s_e(\cdot\,, y, p), H^s_e(\cdot\,, y, p)$ and its far field patterns by $E_{e,\infty}(\cdot\,, y, p)$, $H^s_e(\cdot\,, y, p)$, indicating the dependence on the location y and the polarization p of the dipole. Then we have the mixed reciprocity relation

$$q \cdot E^s(z, d, p) = 4\pi p \cdot E_{e,\infty}(-d, z, q) \tag{5}$$

for all $z \in \mathbb{R}^3 \setminus \bar{D}$, all incident directions $d \in \Omega$, and all polarizations $p \perp d$ and $q \perp d$. For a proof of (5) we refer to Kress [10] or Potthast [15].
By Rellich's lemma, from the coincidence of the far field patterns it follows that the scattered waves for plane wave incidence satisfy $E^s_1(\cdot\,, d, p) = E^s_2(\cdot\,, d, p)$ in the unbounded component G of the complement of $\bar{D}_1 \cup \bar{D}_2$

for all $d \in \Omega$ and $p \perp d$. Then, for scattering of electric dipole fields, from the mixed reciprocity relation (5) we conclude that $E_{1,e,\infty}(\cdot\,, z, q) = E_{2,e,\infty}(\cdot\,, z, q)$ for all $z \in G$ and all polarizations q. Again by Rellich's lemma this implies that for the corresponding scattered waves we have

$$E^s_{1,e}(x, z, q) = E^s_{2,e}(x, z, q) \tag{6}$$

for all $x, z \in G$ and all polarizations q.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. In particular, we have that $z_n := x^* + \frac{1}{n}\,\nu(x^*) \in G$ for sufficiently large n. Then, in view of the well-posedness of the direct scattering problem for the scatterer D_2 (with boundary condition B_2), on one hand we obtain that

$$\lim_{n\to\infty} B_1(E^s_{2,e}(x^*, z_n, q), H^s_{2,e}(x^*, z_n, q)) = B_1(E^s_{2,e}(x^*, x^*, q), H^s_{2,e}(x^*, x^*, q)).$$

On the other hand we find $\lim_{n\to\infty} B_1(E^s_{1,e}(x^*, z_n, q), H^s_{1,e}(x^*, z_n, q)) = \infty$ for $q \perp \nu(x^*)$, because of the boundary condition for $E^s_{1,e}, H^s_{1,e}$ in terms of the electric dipole located at $z_n \to x^*$ for $n \to \infty$. This contradicts (6) and therefore $D_1 = D_2$.

Now, denoting $D = D_1 = D_2$, $E = E_1 = E_2$, and $H = H_1 = H_2$, we assume that we have different boundary conditions $B_1 \neq B_2$. First consider the case where we have impedance boundary conditions with two different continuous impedance functions $\lambda_1 \neq \lambda_2$. Then from $\nu \times H - \lambda_j(\nu \times E) \times \nu = 0$ on ∂D for $j = 1, 2$ we observe that $(\lambda_1 - \lambda_2)\nu \times E = 0$ on ∂D. Therefore for the open set $U := \{x \in \partial D : \lambda_1(x) \neq \lambda_2(x)\}$ we have that $\nu \times E = 0$ and consequently $\nu \times H = 0$ on U. Hence, by Holmgren's uniqueness theorem (see Theorem 2.4 in [9]), $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, i.e., the scattered wave E^s, H^s is an entire solution to the Maxwell equations. This leads to a contradiction as above at the beginning of this section. The case where one of the boundary conditions is the perfect boundary condition can be treated analogously. □

Although there is widespread belief that the far field pattern for one incident plane wave determines the scatterer, establishing this result remains a challenging open problem. Partial progress in this direction was obtained in inverse acoustic obstacle scattering, i.e., for the Helmholtz equation, by Liu[11, 12] and by Yun[17]. Liu showed that if two sound-soft balls have the same far field pattern for one single incident plane wave they must coincide. Yun established the same result for sound-hard balls. We extend their results to electromagnetics giving a simplified proof.

Theorem 2 *A ball and its boundary condition (for constant impedance λ) is uniquely determined by the far field pattern for one incident plane wave.*

Proof. From the explicit solution of the scattering problem (see [10], p.197) for a ball in terms of radiating spherical vector wave functions it can be concluded that the scattered wave has an extension across the boundary into the interior of the ball with the exception of the center. Therefore, two different scattering balls with coinciding far field pattern for one incident plane wave must have the same center, since otherwise the scattered wave would be an entire solution to the Maxwell equation. By symmetry, the electric far far field pattern for scattering of plane waves at a ball centered at the origin satisfies $E_\infty(Q\hat{x}, Qd, Qp) = QE_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in \Omega$, all $p \perp d$, and all rotations Q, i.e., for all orthogonal transformations with $\det Q = 1$. Hence, knowledge of the far field pattern for one incident direction and polarization implies knowledge of the far field pattern for all incident directions and polarizations. Now the statement follows from Theorem 1. □

The scattering by homogeneous penetrable obstacles D leads to transmission problems for the Maxwell equations. Here, uniqueness results for the inverse problems based on the ideas of the proof of Theorem 1 were obtained for scattering from dielectrics by Hähner [6], for scattering from chiral media by Gerlach [5], and for scattering from orthotropic media by Colton, Kress, and Monk [3], by Piana [13] and by Hähner [7].

In addition to the question of uniqueness with one incident field as mentioned above the following open problems are worth mentioning. For acoustic obstacle scattering, Colton and Sleeman [4] showed that given an a priori information on its size a sound-soft scatterer is uniquely determined by the far field of a finite number, depending on the diameter of the scatterer, of incident plane waves. Based on similar ideas as in [4], Stefanov [16] has given a local uniqueness result for sound-soft obstacles with one incident wave. For both cases, corresponding results for electromagnetic waves are not known.

Closely related to local uniqueness is the question of injectivity for the linearization of the inverse problem with respect to the boundary for one incident wave. Here, again the case of sound-soft scatterers in acoustics is settled (see Theoren 5.15 in [2]) whereas nothing is known in electromagnetics.
Numerical methods for the approximate solution of the inverse scattering problem exploiting the above ideas have been suggested and numerically tested by Potthast [14, 15]. The sampling method for solving the inverse problem as suggested by Colton and Kirsch [1] in acoustic scattering and described in [10] for electromagnetic scattering is also related to some extend to these ideas. Roughly speaking, one of the main features of the sampling method is to exploit the fact that an incident field for which the scattered field coincides with the field due to a dipole located at a point z in the interior of the scatterer must become unbounded as z approaches the boundary of the scatterer.

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