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Burst erasure correcting codes for carousel transmission

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Abstract

In several broadcast communication systems, information is repeatedly transmitted with a carousel strategy, i.e., a given codeword is replicated in consecutive frames. When the channel experiences erasure bursts and reception is asynchronous, it becomes important to consider end-around erasure bursts. In this paper, we find the conditions under which a given code is able to correct burst of erasures whenever end-around bursts are taken into consideration. We also describe a constructive algorithm to build a code parity-check matrix with the desired properties.

1 Introduction

In [1], Fossorier shows that, under mild conditions, almost all linear binary codes can be used to correct erasure bursts of length $N-K$, where N is the code length and K is the code dimension, provided that they are suitably interleaved. The only condition that must be satisfied is a full-rank condition for the square matrices in which the columns of the parity-check matrix are partitioned. However, in [1], end-around bursts, i.e., bursts that start at the end of the block and end at the beginning (in a cyclic fashion), are not considered.

End-around burst are important in certain types of applications. In particular, consider a system in which the codeword is repeatedly transmitted in consecutive frames and the reception is asynchronous, so that the reception start is not necessarily aligned with the frame start. In such a case, it may happen that the receiver receives the end of a given frame and the beginning of the following one. Examples of such systems, which are typically broadcast communication systems, are the DVB standard [2] and the GNSSs such as GPS [3] or Galileo [4].

We face in this paper the problem of extending [1], by giving conditions on the parity-check matrix of a binary linear code that is able to correct length-$(N-K)$ erasure bursts, including end-around bursts. Besides considering end-around bursts, we also rephrase the conditions with respect to [1], by using a different concept of linear algebra, i.e., LU-decomposability. Like in [1], we distinguish two cases, according to whether the coding rate is equal to $1-1/m$, for an integer m, or not.

2 Background

Define a length-d *zero run* in a binary vector as a sequence of d consecutive 0's between two 1's, where the run can be cyclic across the vector ends. It is shown in [5] that, if a length-N code has a parity-check equation with a length-d zero run starting at position j, for $j = 1,\ldots,N$, then such code can correct any length-$(d+1)$ erasure burst. The condition can be easily seen to be also necessary. In the following, we give equivalent conditions on the parity-check matrix $\mathbf{H}$ for the code to be able to correct a length-$(N-K)$ erasure burst. First, we write $\mathbf{H}$ in the following form:

$$\mathbf{H} = \left[\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_m, \mathbf{H}'_q\right] \tag{1}$$

where $m = \lfloor 1/(1-r_c) \rfloor$, matrices $\mathbf{H}_1,\ldots,\mathbf{H}_m$ are $(N-K)\times(N-K)$ square matrices, while $\mathbf{H}'_q$ is a $(N-K)\times q$ tall matrix $(0 \le q < N-K)$. Notice that, for $r_c = 1-1/m$, $q = 0$ and the matrix $\mathbf{H}'_q$ disappears.

The following definition, which is commonly found in any linear algebra book, will be useful for our developments.

Definition 2.1 *A full-rank square binary matrix* $\mathbf{A}$ *is said to be LU-decomposable if it can be written as*

$$\mathbf{A} = \mathbf{L}\mathbf{U} \tag{2}$$

where $\mathbf{L}$ *is a lower triangular matrix with ones on the main diagonal and* $\mathbf{U}$ *is an upper triangular matrix with ones on the main diagonal.*

Notice that it is easy to verify whether a given $r\times r$ matrix is LU-decomposable by looking at all its leading principal minors, i.e., the determinants of the $p\times p$ upper-left corners of the matrix, $p = 1,\ldots,r$. If all the leading principal minors of matrix $\mathbf{A}$ are nonzero, then $\mathbf{A}$ is LU-decomposable.

In the next section, we will derive the necessary and sufficient rules that a parity-check matrix must satisfy in order to be able to correct erasure bursts in the context of a carousel transmission.

3 Burst erasure correcting codes

3.1 The case of rate $1-1/m$

In this subsection, we consider the case where the coding rate is $r_c = 1-1/m$, so that the parity-check matrix can be written as

$$\mathbf{H} = [\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_m] \tag{3}$$

Let us define a new parity-check matrix given by

$$\widetilde{\mathbf{H}} = [\mathbf{H}_1\Pi_1, \mathbf{H}_2\Pi_2, \dots, \mathbf{H}_m\Pi_m] \tag{4}$$

where we have applied a column permutation (an interleaver) to each submatrix separately (Π_i is the permutation matrix associated with the column permutation on the i-th submatrix). The following theorem gives the conditions for the code characterized by matrix $\widetilde{\mathbf{H}}$ to be able to correct erasure bursts with length $N-K$.

Theorem 3.1 *Consider an (N,K) code with a $(N-K)\times N$ parity-check matrix $\widetilde{\mathbf{H}}$ written as in (4). Such code can correct a length-$(N-K)$ erasure burst if and only if the following conditions hold:*

- $\mathbf{H}_1, \dots, \mathbf{H}_m$ *are all full rank, and*
- *matrices $\mathbf{M}_i$, $i = 1, \dots, m$, where*

$$\mathbf{M}_i = \begin{cases} \Pi_i{}^{\mathsf{T}}\mathbf{H}_i^{-1}\mathbf{H}_{i+1}\Pi_{i+1}, & i = 1, \dots, m-1 \\ \Pi_m{}^{\mathsf{T}}\mathbf{H}_m^{-1}\mathbf{H}_1\Pi_1, & i = m \end{cases} \tag{5}$$

are all LU-decomposable.

Proof: We first prove the "if" statement. Consider a length-$(N-K)$ burst starting within matrix $\mathbf{H}_i$, for some $i \in \{1, \dots, m\}$. To fix the ideas, let us suppose $i = 1$. If $\mathbf{M}_1$ is LU-decomposable, then we know that

$$\Pi_1{}^{\mathsf{T}}\mathbf{H}_1^{-1}\mathbf{H}_2\Pi_2 = \mathbf{L}_1\mathbf{U}_1 \tag{6}$$

where $\mathbf{L}_1$ is lower-triangular, $\mathbf{U}_1$ is upper-triangular and both are invertible. Thus, another parity-check matrix for the same code is given by

$$\widetilde{\mathbf{H}}_1 = \mathbf{L}_1^{-1}\Pi_1{}^{\mathsf{T}}\mathbf{H}_1^{-1}\widetilde{\mathbf{H}} = [\mathbf{L}_1^{-1}, \mathbf{U}_1, \mathbf{H}'_3, \dots, \mathbf{H}'_m] \tag{7}$$

where $\mathbf{H}'_i = \Pi_1{}^{\mathsf{T}}\mathbf{H}_1^{-1}\mathbf{H}_i\Pi_i$, for $i = 3, \dots, m$. Since $\mathbf{L}_1$ is lower triangular and $\mathbf{U}_1$ is upper triangular, there is a length-$(N-K)$ zero-run starting at every position between 1 and $(N-K)$. Thus, the erasure burst is correctable.

We then pass to the "only if" statement. (Again, we consider only bursts starting within the first $N-K$ positions, since the generalization is straightforward.) Suppose that the code corrects every length-$(N-K)$ burst. Then, for $j = 2, \dots, N-K$, there will be a parity-check equation with a length-$(N-K-1)$ zero run starting at position j. Thus, there exists a parity-check matrix for the code with a structure like in (7). This implies that *any* parity-check matrix of the code written as in (4) has $\mathbf{H}_1$ and $\mathbf{H}_2$ full rank and that $\mathbf{M}_1$ is LU-decomposable. ■

Without considering the end-around burst, it is proved in [1] that there exists a set of interleavers $\{\Pi_i\}$, $i = 1, \dots, m$ that meet the conditions of Theorem 3.1 for every matrix $\mathbf{H}$ for which $\mathbf{H}_1, \dots, \mathbf{H}_m$ are full rank. We could not be able to prove that this also holds when introducing the end-around burst. However, we have not found any counterexample either.

3.2 The case of arbitrary rate

In this section, we generalize Theorem 3.1 to the case of arbitrary rate. Notice that, when $r_c \neq 1-1/m$, the matrix $\mathbf{H}'_q$ in (1) must be taken into account. Thus, the interleaved version of matrix (1) will be given by

$$\widetilde{\mathbf{H}} = \left[\mathbf{H}_1\Pi_1, \mathbf{H}_2\Pi_2, \dots, \mathbf{H}_m\Pi_m, \mathbf{H}'_q\Pi'_q\right] \tag{8}$$

where Π'_q is a size-q permutation matrix. Define also $\mathbf{H}_1 = [\mathbf{H}_1^L, \mathbf{H}_1^R]$, where $\mathbf{H}_1^L$ contains the first $N-K-q$ columns of $\mathbf{H}_1$, and $\mathbf{H}_1^R$ the remaining q. In order to keep the conditions simple, we only give a sufficient condition, instead of a necessary and sufficient condition as in the previous subsection.

Theorem 3.2 *Consider an (N,K) code with a $(N-K)\times N$ parity-check matrix $\widetilde{\mathbf{H}}$ written as in (8). Such code can correct a length-$(N-K)$ erasure burst if the following conditions hold:*

- $\mathbf{H}_1, \dots, \mathbf{H}_m$, *are all full rank,*
- *the matrix $\mathbf{H}_{qL} = [\mathbf{H}'_q, \mathbf{H}_1^L]$ is full rank,*
- Π_1 *does not mix the columns of $\mathbf{H}_1^L$ and $\mathbf{H}_1^R$, i.e.,*

$$\Pi_1 = \begin{bmatrix} \Pi_1^L & \mathbf{0} \\ \mathbf{0} & \Pi_1^R \end{bmatrix}$$

(where Π_1^L and Π_1^R are permutation matrices with sizes $N-K-q$ and q, respectively),

- *matrices $\mathbf{M}_i$, $i = 1, \dots, m$, where*

$$\mathbf{M}_i = \begin{cases} \Pi_i{}^{\mathsf{T}}\mathbf{H}_i^{-1}\mathbf{H}_{i+1}\Pi_{i+1}, & i = 1, \dots, m-1 \\ \Pi_m{}^{\mathsf{T}}\mathbf{H}_m^{-1}\mathbf{H}_{qL}\begin{bmatrix} \Pi'_q & \mathbf{0} \\ \mathbf{0} & \Pi_1^L \end{bmatrix}, & i = m \\ \Pi'_q{}^{\mathsf{T}}[\mathbf{I}_q|\mathbf{0}]\mathbf{H}_{qL}^{-1}\mathbf{H}_1^R\Pi_1^R, & i = m+1 \end{cases} \tag{9}$$

are all LU-decomposable.

Proof: The proof is similar to the proof of Theorem 3.1. The conditions on $\mathbf{M}_m$ and on $\mathbf{M}_{m+1}$ guarantee the correctability of end-around bursts. ■

Algorithm 1: Interleaver search for $r_c = 1 - 1/m$.

```
input  : Matrices $\mathbf{A}_1,\dots,\mathbf{A}_m$, where $\mathbf{A}_i = \mathbf{H}_i^{-1}\mathbf{H}_{i+1}$ and $\mathbf{A}_m = \mathbf{H}_m^{-1}\mathbf{H}_1$
output: Permutations $\pi_i$, $i = 1,\dots,m$
IndexVec = [];
foreach $i \in [1,m]$ do
    $\mathbf{M}_i^{(0)} = \mathbf{A}_i$;
    $\pi_i^{(0)} = []$;
    IndexVec = [IndexVec; $(1,\dots,n)$];
$[\pi_1,\dots,\pi_m]$ = permustep ($\mathbf{M}_1^{(0)},\dots,\mathbf{M}_m^{(0)},\pi_1^{(0)},\dots,\pi_m^{(0)}$, IndexVec );
```

Algorithm 2: Algorithm permustep

```
input  : Matrices $\mathbf{A}_1,\dots,\mathbf{A}_m$, input permutations $\pi_i^I,\dots,\pi_m^I$, IndexVec
output: Output permutations $\pi_i^O$, $i = 1,\dots,m$
1  $n$ = size(IndexVec, 2);
2  if ($n == 1$) then
3      foreach $i \in [1,m]$ do
4          $\pi_i^O = [\pi_i^I$, IndexVec$(i,1)]$;
5  else
6      foreach $m-$tuple $(j_1,\dots,j_m) \in \{1,\dots,n\}^m$ do
7          $\mathbf{M}_i^{(\mathrm{app})} = \mathbf{A}_i([1,\dots,n]\backslash j_i, [1,\dots,n]\backslash j_{i+1})$, $i = 1,\dots,m-1$;
8          $\mathbf{M}_m^{(\mathrm{app})} = \mathbf{A}_m([1,\dots,n]\backslash j_m, [1,\dots,n]\backslash j_1)$;
9          if $\mathbf{M}_i^{(\mathrm{app})}$, $i = 1,\dots,m$ are full $-$ rank then
10             foreach $i \in [1,m]$ do
11                 $\pi_i^O = [\pi_i^I$, IndexVec$(i, j_i)]$;
12                 IndexVecApp$(i,:)$ = IndexVec$(i,[1:n]\backslash j_i)$;
13                 $[\pi_1^O,\dots,\pi_m^O]$ =
                   permustep($\mathbf{M}_1^{(\mathrm{app})},\dots,\mathbf{M}_m^{(\mathrm{app})},\pi_1^O,\dots,\pi_m^O$, IndexVecApp);
14 return $\pi_i^O$, $i = 1,\dots,m$
```

It is clear that the conditions of Theorem 3.2 are not necessary. Just to make an example, there might be a code correcting a length-$(N-K)$ erasure burst for which the interleaver Π_1 mixes the columns of $\mathbf{H}_1^L$ and $\mathbf{H}_1^R$. (However, this case can be included in the theorem by a suitable redefinition of matrix $\mathbf{H}_1$.) Again, we could not prove that, for any matrix $\mathbf{H}$ satisfying all conditions of Theorem 3.2 on matrices $\mathbf{H}_1,\dots,\mathbf{H}_m,\mathbf{H}_q'$, it is always possible to derive a set of interleavers that satisfy the conditions on matrices $\mathbf{M}_1,\dots,\mathbf{M}_{m+1}$.

In the next section, we describe a practical algorithm to derive, for a parity-check matrix $\mathbf{H}$ that satisfies the conditions on submatrices $\mathbf{H}_1,\dots,\mathbf{H}_m,\mathbf{H}_q'$, the set of interleavers.

4 Algorithmic approach

In order to enforce joint LU-decomposability of matrices $\mathbf{M}_1,\dots,\mathbf{M}_m$ (and $\mathbf{M}_{m+1}$ if $q > 0$), we have implemented a tree-search algorithm in which a node of depth p, $p = 1,\dots,N-K$ represents a possible joint choice of the p-th columns of matrices $\Pi_1,\dots,\Pi_m,\Pi_q'$. The algorithm adopts a depth-first search of the tree until it reaches a depth-$(N-K)$ leaf.

For the case $q = 0$, the pseudocode is given in Alg. 1, which points to a recursive function `permustep`, which is the core of the algorithm and whose pseudocode is represented in Alg. 2. Since exploring all possible choices may entail large execution times, a possible option that was seen to be efficient is to explore first those choices that involve a small number of ones in the rows and columns of $\mathbf{A}_i$, $i = 1,\dots,m$ (see lines 7-8 in Alg. 2). A maximum number of visited nodes can also be set in order to avoid very long simulation times, after which a failure is declared. The algorithm for $q > 0$ is based on the same concept.

Although we do not have formal guarantees on the execution time of the algorithm, we have obtained a correct solution without much computational effort for all the practical cases we have tested.

References

[1] Fossorier, M., "Universal Burst Error Correction," *IEEE International Symposium on Information Theory (ISIT)*, Seattle, USA, 2006, pp. 1969–1973.

[2] ETSI, "Digital Video Broadcasting (DVB); DVB specification for data broadcasting," *ETSI EN 301 192 v1.5.1*, 2016.

[3] Kaplan, E. D., and Hegarty C. J. (Eds.), "Understanding GPS - Principles and Applications, II Ed.," *Artech House*, Boston, 2006.

[4] European Union, "Signal-In-Space Interface Control Document," Issue 1.3, December 2016.

[5] Tai, Y. Y., Zeng, L., Lan, L., Song, S., and Lin, S., "Algebraic Construction of Quasi-Cyclic LDPC Codes - Part II: For AWGN and Binary Random and Burst Erasure Channels," *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, LNCS, Vol. 3857, Feb. 2006.