Some Features of Electromagnetic Wave Scattering from Invisible Spherical Lens with Negative Refractive Index

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Abstract

Scattering of a plane wave from a spherically symmetric lens with negative refractive index, invisible from the viewpoint of geometrical optics, is analyzed numerically using the hybrid projection method. It is detected that, unlike the invisible lens with positive refractive index, the minima of the radar cross section in forward direction corresponding to the invisibility take place at integer values of ka where k is the wavenumber and a is the lens radius, instead of half-integer values, and that the indicated minima are not so deep because of the astigmatism in focusing of the rays at the lens axis.

1. Introduction

One of the extensively developing contemporary scientific areas associated with metamaterials is transformation electromagnetics aimed in particular at creation of covers or shells making objects invisible. In the indicated area there have been proposed several types of invisible cloaks and a so-called invisible gradient-index lens [1, 2] made of isotropic material with positive refractive index depending only on radial coordinate and approaching infinity in the center. The rays enter the lens without reflection, make loops around the center, and exit the lens also without reflection in the same direction as they entered. Another modification of the invisible lens based, unlike [1, 2], on material with negative refractive index has recently been proposed in [3] where the author has derived expressions for calculation of the refractive index profile and ray trajectories.

The lenses indicated above can be invisible from the viewpoint of the laws of geometrical optics. However geometrical optics is an approximate theory, and therefore it is of interest to study wave scattering from the invisible lenses at rigorous statement of the problem. Such studies of the invisible lens with positive refractive index have been carried out in [4, 5] using the WKB method and commercial software COMSOL.

The purpose of the present work is numerical analysis of plane electromagnetic wave scattering from the invisible spherical lens with negative refractive index in rigorous statement of the problem and comparison of the scattering performance with similar results corresponding to the invisible lens with positive refractive index.

2. Geometrical-Optics Characteristics

As derived in [3], the refractive index magnitude of the invisible lens is determined by

$$n(r) = \frac{4}{\left[(s + r/a)^{2/3} + (s - r/a)^{2/3} + 1/3\right]^2} \quad (1)$$

where $s = [(r/a)^2 + 1/27]^{1/2}$ and a is the lens radius. Analysis of (1) shows that the refractive index magnitude equals 1 on the lens surface and 4 in the lens center. The ray trajectories in the lens can be calculated using formula

$$\theta(r) = -\frac{3\pi}{2} - \arccos\frac{ah}{rn(r)} + 2\arcsin\frac{\sqrt{n(r) - h^2}}{\sqrt{n(r)}\sqrt{1 - h^2}} \quad (2)$$

where $h = \sin\psi$ and ψ is the angular coordinate of the point on the lens surface where a ray enters the lens. Some examples of the ray trajectories are shown in Figure 1.
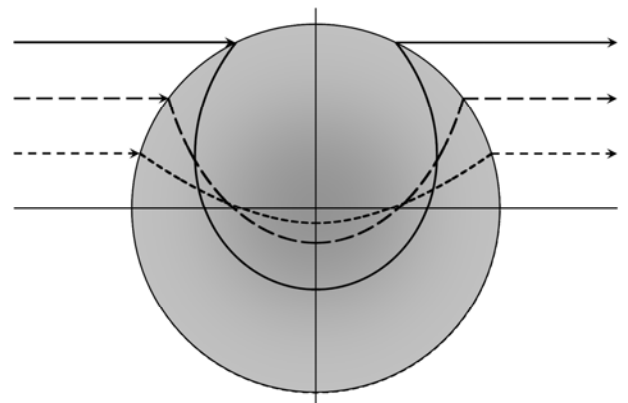

Figure 1. Ray trajectories in the invisible lens with negative refractive index.

As we see, each ray entering the lens experience negative refraction, makes an arc below the lens center (or above the lens center if the ray enters the lens below its axis), experiences negative refraction at the exit point, and then propagates in the direction that would take place in the absence of the lens. Note that unlike the lens with positive refractive index, the trajectories in Figure 1 have no self-intersections, and this feature corresponds to finite limits of the refractive index variation.

3. Formulation of the Problem of Scattering

The geometry of the problem is shown in Figure 2 where a sphere of radius a, relative permittivity $\varepsilon(r)$, and constant relative permeability μ, is assumed to be illuminated by a plane electromagnetic wave of unit amplitude polarized along the x axis and propagating along the z axis. The electric and magnetic field strengths of the incident wave are determined by formulas $\mathbf{E}^i = \mathbf{e}_x e^{ikz}$ and $\mathbf{H}^i = (\mathbf{e}_y/\eta)e^{ikz}$, $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors, η is the wave resistance of free space, $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength in free space. Dependence of the fields on time assumed to be taken in the form $e^{-i\omega t}$ here and below is suppressed.

To solve the problem we represent the transverse components of the fields as expansions in terms of spherical TE and TM waves [6]

$$\mathbf{E}_{1\tau} = \frac{1}{kr} \sum_{l=1}^{\infty} \{[A_l \psi_l(kr) + R_{1l}\zeta_l(kr)]\mathbf{T}_{1l}^e$$

$$- i[A_l \psi_l'(kr) + R_{2l}\zeta_l'(kr)]\mathbf{T}_{2l}^e\}, \qquad (3)$$

$$\mathbf{H}_{1\tau} = \frac{1}{\eta kr} \sum_{l=1}^{\infty} \{-i[A_l \psi_l'(kr) + R_{1l}\zeta_l'(kr)]\mathbf{T}_{1l}^h$$

$$+ [A_l \psi_l(kr) + R_{2l}\zeta_l(kr)]\mathbf{T}_{2l}^h\}, \qquad (4)$$

$$\mathbf{E}_{2\tau} = \sum_{l=1}^{\infty} [E_{1l}(r)\mathbf{T}_{1l}^e + E_{2l}(r)\mathbf{T}_{2l}^e], \qquad (5)$$

$$\mathbf{H}_{2\tau} = \frac{1}{\eta} \sum_{l=1}^{\infty} [H_{1l}(r)\mathbf{T}_{1l}^h + H_{2l}(r)\mathbf{T}_{2l}^h], \qquad (6)$$

where $A_l = i^l[(2l+1)2\pi]^{1/2}$ are coefficients of expansion corresponding to the incident wave, R_{jl} are unknown constant coefficients for the TE ($j=1$) and TM ($j=2$) waves, $E_{jl}(r)$ and $H_{jl}(r)$ are unknown variable coefficients, $\psi_l(kr)$ and $\zeta_l(kr)$ are functions of Riccati-Bessel and Riccati-Hankel, and $\mathbf{T}_{jl}^e$, $\mathbf{T}_{jl}^h$ are orthonormalized transverse spherical functions [6] of angles θ and φ.

Projection of the Maxwell's equations written for the medium in the lens on the transverse spherical functions for each l results in two independent pairs of ordinary differential equations
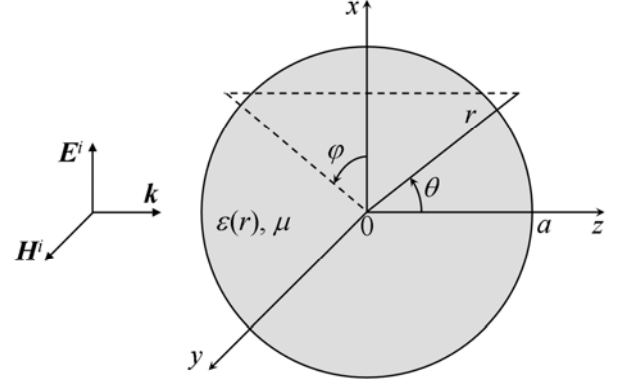

Figure 2. Geometry of the problem of scattering.

$$ik \frac{dV_1}{dr} + \left[k^2 \varepsilon(r) - \frac{l(l+1)}{r^2 \mu}\right]U_1 = 0, \qquad (7)$$

$$V_1 = \frac{1}{ik\mu} \frac{dU_1}{dr}, \qquad (8)$$

$$ik \frac{dU_2}{dr} + \left[k^2 \mu - \frac{l(l+1)}{r^2 \varepsilon(r)}\right]V_2 = 0, \qquad (9)$$

$$U_2 = \frac{1}{ik\varepsilon(r)} \frac{dV_2}{dr}, \qquad (10)$$

with respect to functions $U_j(r) = krE_{jl}(r)$, $V_j(r) = krH_{jl}(r)$. For obtaining solution of the equations given above, we represent functions U_1 и V_2 as

$$U_1(r) = \sum_{n=1}^{N} U_{1n} f_n(r), \qquad (11)$$

$$V_2(r) = \sum_{n=1}^{N} V_{2n} f_n(r), \qquad (12)$$

where U_{1n} and V_{2n} are unknown constant coefficients, $f_n(r)$ are so-called triangular functions [6] with the tops located at $r_n = n\Delta$, $\Delta = a/N$, and N is the number of segments. Projection of (7) and (9) on $f_n(r)$, account for (8) and (10), as well as matching of electric fields (3) and (5) and magnetic fields (4) and (6) on the lens surface allow reducing the problem to systems of linear algebraic equations

$$\sum_{n=1}^{N} Z_{n'n}^{(1)} U_{1n} + R_{1l}\zeta_l'(ka)\delta_{n'N} = -A_l \psi_l'(ka)\delta_{n'N}, \qquad (13)$$

$$U_{1N} - R_{1l}\zeta_l(ka) = A_l \psi_l(ka) \qquad (14)$$

for TE waves and

$$\sum_{n=1}^{N} Z_{n'n}^{(2)} V_{2n} + R_{2l}\zeta_l'(ka)\delta_{n'N} = -A_l\psi_l'(ka)\delta_{n'N}, \qquad (15)$$

$$V_{2N} - R_{2l}\zeta_l(ka) = A_l\psi_l(ka) \qquad (16)$$

for TM waves where

$$Z_{n'n}^{(1)} = \int_0^a f_{n'}f_n\left[k\varepsilon - \frac{l(l+1)}{kr^2\mu}\right]dr - \frac{1}{k\mu}\int_0^a f_{n'}'f_n'dr, \qquad (17)$$

$$Z_{n'n}^{(2)} = \int_0^a f_{n'}f_n\left[k\mu - \frac{l(l+1)}{kr^2\varepsilon}\right]dr - \frac{1}{k}\int_0^a \frac{1}{\varepsilon}f_{n'}'f_n'dr \qquad (18)$$

are matrix elements.

Coefficients R_{1l} and R_{2l}, corresponding to the scattered field, determined as a result of solution of systems (13) and (14) as well as (15) and (16) are used for calculation of the bi-static radar cross section (RCS)

$$\sigma(\theta,\varphi) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}^s|^2}{|\mathbf{E}^i|^2}$$

$$= \frac{\lambda^2}{\pi}\left|\sum_{l=1}^{L}(-i)^l(R_{1l}\mathbf{T}_{1l}^e + R_{2l}\mathbf{T}_{2l}^e)\right|^2 \qquad (19)$$

obtained with use of incident and scattered fields in (3) as well as asymptotic expressions

$$\zeta_l(kr) \approx (-i)^{l+1}e^{ikr}, \quad \zeta_l'(kr) \approx (-i)^l e^{ikr}$$

for Riccati-Hankel functions and their derivatives at large kr and accounting for finite number L of spherical harmonics for each type.

4. Numerical Results

The method described above has been implemented in two MATLAB codes corresponding to the lens with negative refractive index and to the lens with positive refractive index. Integrals in (17) and (18) for the terms which do not involve $\varepsilon(r)$ have been calculated using analytical formulas derived for them, while the integrals for the terms containing $\varepsilon(r)$ have been calculated numerically. In case of the lens with positive refractive index the calculations have been carried out with extraction and analytical integration of the singularity in the lens center [2, 3].

The relative permittivity of the lens with positive refractive index is supposed to be determined by the square of the refractive index given in [2, 3], while the relative permeability is $\mu = 1 + i\mu''$ where μ'' determines possible loss in the lens. As for the lens with negative refractive index, its permittivity is determined as $\varepsilon(r) = -n^2$

where n is determined by (1), and the relative permeability is $\mu = -1 + i\mu''$.

Like it takes place in [6], the code operation is validated by the accuracy of fulfilling the optical theorem in case of lossless lenses [7], as well as by convergence of the results with increasing the number of nodes N and the number of retained spherical waves L of each type.

Calculations of RCS (19) have shown that the level of back scattering at $\theta = \pi$ for both positive and negative refractive indexes is very low for any lens radius. Therefore the indicated parameter can not appropriately characterize the lens from the viewpoint of its invisibility. More demonstrative parameter is the level of (19) in forward direction at $\theta = 0$. Dependences of the indicated parameter normalized to πa^2 in dB, i.e. $10\lg[\sigma(0,0)/(\pi a^2)]$, on ka for the invisible lenses with negative and positive refractive indexes are presented in Figure 3. The results have been obtained at node spacing $\Delta = 0.005\lambda$. The number of retained Fourier harmonics L corresponds to rounded values of $2ka+5$.

Comparing the curves in Figure 3, we see that location of the minima for the lens with negative refractive index does not coincide with location of the minima for the lens with positive refractive index. As explained in [3], the location of the minima in the latter case is determined by an additional phase shift of $\pi/2$ multiplied by 2 because of focusing of the rays in line on the lens axis. Unlike that, the rays in the lens with negative index (Figure 1) experience point (not line) focusing on the axis two times though with astigmatism. Each such focusing results in phase shift of π [8], so the total phase shift becomes 2π and the minima turn out to be located between the minima for the lens with positive refractive index.
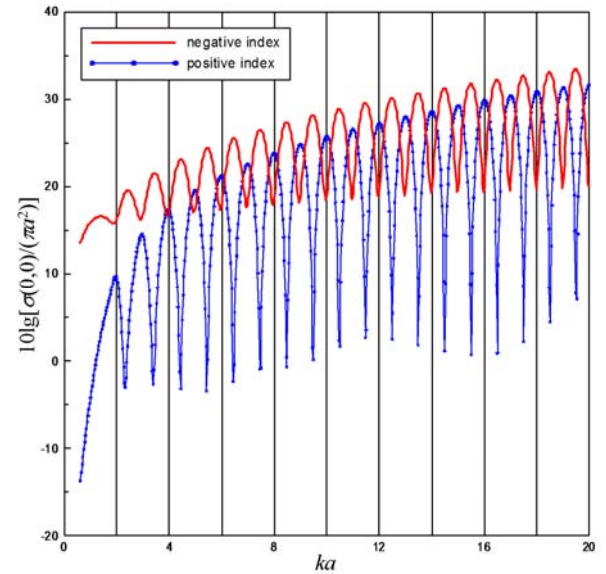

Figure 3. RCS in the forward direction for the lossless invisible lenses with negative and positive refractive indexes.
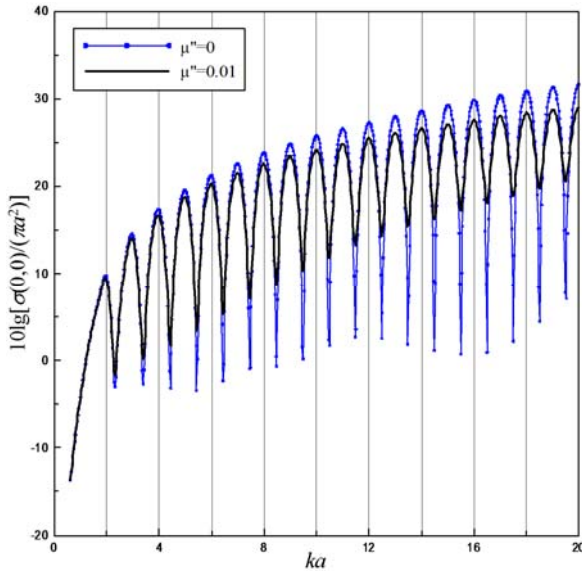
Figure 4. RCS in the forward direction for the lossy and lossless invisible lens with positive refractive index.

Figure 5. RCS in the forward direction for the lossy and lossless invisible lens with negative refractive index.

One more feature of the lens with negative refractive index is that the RCS minima shown in Figure 3 have a considerably higher level compared to the case of the lens with positive refractive index. The indicated feature is just explained by the presence of the astigmatism mentioned above owing to which we observe the filling of the RCS nulls.

It is also of interest to study the influence of losses in the lenses on the effect of invisibility. Some results of such a study are presented in Figure 4 for the case of the lens with positive refractive index and in Figure 5 for the case of the lens with negative refractive index. As it could be expected the losses in the invisible lenses also result in additional filling the nulls of the RCS in forward direction.

5. Acknowledgements

6. References

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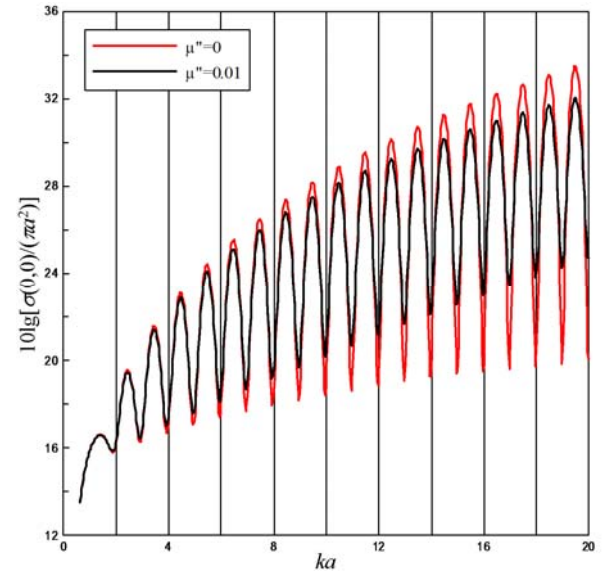
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