Power Maximization for a Multiport Network Described by the Admittance Matrix

Ben Minnaert, Giuseppina Monti, Alessandra Costanzo, and Mauro Mongiardo

Abstract — This work analyzes a general (reciprocal or nonreciprocal) multiport system described by its admittance matrix. The general case of a link using a multiple-input and multiple-output configuration is solved by determining the optimal loads for maximizing the total power delivered to the loads. As a specific application of interest, the proposed theory is demonstrated on a wireless link based on capacitive coupling.

1. Introduction

Consider a multiport network with an arbitrary number of input and output ports. The input ports are fed by power supplies. The purpose of this paper is to propose an easy procedure to determine the terminations of the multiport network that maximize the power transfer to the output ports.

The multiport network is considered as a black box and can be any physical network. In this work, focus is on multiports for which it is convenient to fully characterize the multiport by its admittance matrix. Application examples include the following:

- Coupled transmission lines, e.g., multiconductor transmission lines in power systems for efficient transmission of electrical energy from one point to another [1, 2].
- Any form of wireless power transfer (WPT) systems such as microwave/RF WPT or near-field WPT. In particular capacitive WPT can be easier described by its admittance matrix [3], contrary to an inductive WPT multiport, which is more manageable if characterized by its impedance matrix [4].
- Propagation channels of multiple-input multiple-output (MIMO) networks [5].
- Microwave filters and couplers [6, 7].
- Nonreciprocal circuit networks such as ferrite isolators [7].

This work is an extension on [8] that described the method for a reciprocal wireless power transfer system. In the current work, we emphasize that the method is applicable to any reciprocal or nonreciprocal multiport network. Compared to [8], it now includes the calculation of the maximum output power, corresponding input power, and system efficiency (power gain) of the network. Moreover, the numerical verification is more elaborated by explicitly adding the admittance matrix and the simulation circuit in SPICE of a capacitive multiport example.

2. Multiport Representation With M Input and N Output Ports

A multiport network $N'$ with $M$ input and $N$ output ports is considered and fully characterized by its admittance matrix $\mathbf{Y}$. The $M$ input ports of the network are connected to $M$ current sources (Figure 1, left side). At the $N$ output ports, loads admittances $Y_{L,i} = G_{L,i} + jB_{L,i}$ are present ($i = 1, \ldots, N$), with $G_{L,i}$ and $B_{L,i}$ the load conductance and load susceptance, respectively (Figure 1, right side). The peak current and voltage phasors at the $M$ input and $N$ output ports, respectively, are defined in Figure 1. The following matrices are introduced:

$$
\mathbf{i}_M = \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_M
\end{bmatrix}, \mathbf{v}_M = \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_M
\end{bmatrix}, \mathbf{i}_N = \begin{bmatrix}
I_{L,1} \\
I_{L,2} \\
I_{L,3} \\
\vdots \\
I_{L,N}
\end{bmatrix}, \mathbf{v}_N = \begin{bmatrix}
V_{L,1} \\
V_{L,2} \\
V_{L,3} \\
\vdots \\
V_{L,N}
\end{bmatrix}
$$

The relation between the voltages and the currents of the multiport can be described by

$$
\begin{bmatrix}
\mathbf{i}_M \\
\mathbf{i}_N
\end{bmatrix} = \begin{bmatrix}
Y_{MM} & Y_{MN} \\
Y_{NM} & Y_{NN}
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_M \\
\mathbf{v}_N
\end{bmatrix}
$$

where the admittance matrix of the network $N'$ has been partitioned into four submatrices $Y_{MM}$, $Y_{MN}$, $Y_{NM}$, and $Y_{NN}$. The subscripts of the submatrices indicate their dimension.

By applying the Norton’s theorem, for the multiport network it is possible to derive the equivalent circuit illustrated in Figure 2, where the currents $I_n^{(no)}$ are the Norton currents ($i = 1, \ldots, N$). Notice that the $M$ input ports are replaced by open circuits. The $N$ loads of the receiver are represented by the network $N_L$, described by the admittance matrix $Y_L$. 

Manuscript received 26 August 2020.
Ben Minnaert is with the Odisee University College of Applied Sciences, Ghent, Belgium; ben.minnaert@odisee.be.
Giuseppina Monti is with the Department of Engineering for Innovation, University of Salento, Lecce, Italy; giuseppina.monti@unisalento.it.
Alessandra Costanzo is with the Department of Electrical, Electronic and Information Engineering “Guglielmo Marconi,” University of Bologna, Bologna, Italy; alessandra.costanzo@unibo.it.
Mauro Mongiardo is with the Department of Engineering, University of Perugia, Perugia, Italy; mauro.mongiardo@unipg.it.
The Norton currents can be derived by short-circuiting the output ports \( v_N = 0 \) and equal the following [8]:

\[
\mathbf{i}_N = \frac{Y_{NM} \mathbf{Y}_{MM}^{-1}}{C_0} \mathbf{i}_M = \begin{bmatrix} \mathbf{i}^{(no)}_1 \\ \mathbf{i}^{(no)}_2 \\ \vdots \\ \mathbf{i}^{(no)}_N \end{bmatrix}
\]  

\[ (3) \]

The Norton admittance matrix \( \mathbf{Y}_0 \), which characterizes network \( N_0 \), is defined by

\[
\mathbf{i}_N = \mathbf{Y}_0 \cdot \mathbf{v}_N
\]

\[ (4) \]

It can be expressed as function of the elements of the original admittance matrix \( \mathbf{Y} \) of the multiport [8]:

\[
\mathbf{Y}_0 = \mathbf{Y}_{NN} - \mathbf{Y}_{NM} \cdot \mathbf{Y}_{MM}^{-1} \cdot \mathbf{Y}_{MN}
\]

\[ (5) \]

### 3. Optimal Loads for Power Maximization

The goal of this work is to determine the loads that realize maximum power transfer from the \( M \) input ports to the \( N \) output ports, i.e., that maximize the total output power \( P_{out} \), defined as

\[
P_{out} = \sum_{i=1}^{N} P_i
\]

\[ (6) \]

with \( P_i \) the output power delivered to load \( Y_{L,i} \). By applying Norton’s theorem, the original circuit of Figure 1 was replaced by the equivalent circuit of Figure 2 with Norton admittance matrix \( \mathbf{Y}_0 \) and current sources \( \mathbf{i}^{(no)} \).

Spinei [9] determined a generalized condition for the voltage-current relations at the ports of a multiport network for achieving maximum power transfer to passive loads:

\[
\mathbf{v}_N = (\mathbf{Y}_0 + \mathbf{Y}_0^+)^{-1} \mathbf{i}^{(no)}
\]

\[ (7) \]

with \( \mathbf{Y}_0^+ \) the conjugate transpose of \( \mathbf{Y}_0 \). For a reciprocal network the conjugate transpose coincides with the conjugate. The procedure described in this work is also valid for nonreciprocal networks.

Since \( \mathbf{Y}_0 \) and \( \mathbf{i}^{(no)} \) are known from the previous section, the optimal voltages \( \mathbf{v}_N \) at the \( N \) output ports that realize maximum power transfer are known.

Substituting (7) into the network equations of Figure 2 results in the optimal currents [8]:

\[
\mathbf{i}_N = \mathbf{i}^{(no)} - \mathbf{Y}_0 (\mathbf{Y}_0 + \mathbf{Y}_0^+)^{-1} \mathbf{i}^{(no)}
\]

\[ (8) \]

Equations (7) and (8) determine the voltages and currents at the loads for the maximum power configuration. The load admittances that realize maximum power transfer are therefore given by

\[
Y_{L,i} = G_{L,i} + jB_{L,i} = \frac{i_{L,i}}{P_i}
\]

\[ (9) \]

where \( i = 1, \ldots, N \), with \( V_{L,i} \) and \( I_{L,i} \) the elements of \( \mathbf{v}_N \) and \( \mathbf{i}_N \), respectively, as described by (7) and (8).

The procedure to find the loads for power maximization for a multiport network can be summarized as follows:

1) Establish (e.g., by measurement or simulation) the admittance matrices \( \mathbf{Y}_{MM}, \mathbf{Y}_{MN}, \mathbf{Y}_{NM}, \) and \( \mathbf{Y}_{NN} \) of the multiport network given by (2).

2) Determine the Norton current sources \( \mathbf{i}^{(no)}_N \) from (3).

3) Set up the Norton admittance matrix \( \mathbf{Y}_0 \) using (5).

4) Calculate the voltages \( \mathbf{v}_N \) (see (7)) and currents \( \mathbf{i}_N \) (see (8)) for the loads at the maximum power configuration.

5) Determine the optimal loads \( Y_{L,i} \) from (9).

This procedure maximizes the power output \( P_{out} \) as defined in (6). Its value can be calculated by
The theory presented in the previous section is valid for any multiport network, regardless of the total number of \(M + N\) ports and its specific implementation; in order to be applied, the presented methodology just requires the experimental or analytical derivation of the admittance matrix of the network. In this section, the procedure is demonstrated and numerically verified on an example application: a capacitive WPT system with two transmitters \((M = 2)\) and three receivers \((N = 3)\). The approximated equivalent circuit of the analyzed link is illustrated in Figure 3 [10].

At the two input ports (Figure 3, left side), two current sources \(I_1\) and \(I_2\) power the system. At the three output ports (Figure 3, right side), load admittances \(Y_{L1}, Y_{L2},\) and \(Y_{L3}\) are connected. The voltages \(V_j\) and currents \(I_j\) at the ports are defined in the figure \((j = 1, \ldots, 5)\).

The shunt conductances \(g_{ij}\) describe the losses in the circuit. The mutual capacitances \(C_{13}, C_{14}, C_{15}, C_{23}, C_{24},\) and \(C_{25}\) represent the desired electric coupling between the transmitter capacitances \(C_1, C_2,\) and the receiver capacitances \(C_3, C_4, C_5\). They realize the wireless link between transmitters and receivers [6]. Undesired electric coupling is present between both transmitters, indicated by the mutual capacitance \(C_{12}\). An undesired coupling also is present between the receivers: \(C_{34}, C_{35},\) and \(C_{45}\).

To obtain a resonant scheme, the inductors \(L_j = 1/\omega_0^2C_j\) are added with \(\omega_0\) the operating angular frequency of the current sources \(I_1\) and \(I_2\). The coupling factor \(k_{ij}\) is defined as \(k_{ij} = C_{ij}/\sqrt{C_iC_j}(i, j = 1, \ldots, 5)\).

As an example, consider the system with numerical values indicated in Table 1 and operating at \(f_0 = 10\) MHz. The entire multiport system (indicated by the dashed rectangle in Figure 3) is fully determined by the admittance matrix \(Y\), which is, at the resonance angular frequency \(\omega_0\), equal to

\[
Y = \begin{bmatrix} Y_{MM} & Y_{MN} \\ Y_{NN} \\ \end{bmatrix} = \begin{bmatrix} 1.00 & -2.04j & -5.58j & -4.41j & -3.32j \\ -2.04j & 1.25 & -4.30j & -3.26j & -2.31j \\ -5.58j & -4.30j & 1.50 & -0.745j & -0.281j \\ -4.41j & -3.26j & -0.745j & 1.75 & -0.666j \\ -3.32j & -2.31j & -0.281j & -0.666j & 2.0 \end{bmatrix} \text{mS}
\]

with nondiagonal elements \(\omega_0C_{ij}\). The coupling factors between the transmitters and receivers are indicated in Table 2.

By using (9), the values reported in Table 3 can be obtained for the optimal terminating admittances. The optimal load susceptances are negative, i.e., they correspond to shunt inductors whose values are reported in the table.

To verify the analytical results summarized in Table 3 circuitual simulations have been performed in SPICE. The equivalent pi circuit for coupled capacitors, valid near resonance, is applied (Figure 4). The simulations are executed in the time domain with maximum time step 1 ps. First, a simulation with the network terminated on the optimal admittances given in

---

**Table 1.** Given circuit parameters of the example capacitive WPT system

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g_{11})</td>
<td>1.00 mS</td>
<td>(C_1)</td>
<td>350 pF</td>
</tr>
<tr>
<td>(g_{22})</td>
<td>1.25 mS</td>
<td>(C_2)</td>
<td>300 pF</td>
</tr>
<tr>
<td>(g_{33})</td>
<td>1.50 mS</td>
<td>(C_3)</td>
<td>250 pF</td>
</tr>
<tr>
<td>(g_{44})</td>
<td>1.75 mS</td>
<td>(C_4)</td>
<td>225 pF</td>
</tr>
<tr>
<td>(g_{55})</td>
<td>2.00 mS</td>
<td>(C_5)</td>
<td>200 pF</td>
</tr>
<tr>
<td>(I_1)</td>
<td>100 mA</td>
<td>(f_0)</td>
<td>10 MHz</td>
</tr>
<tr>
<td>(I_2)</td>
<td>200 mA</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

**Table 2.** Coupling factors of the analyzed numerical example

<table>
<thead>
<tr>
<th>Desired coupling</th>
<th>Value (%)</th>
<th>Undesired coupling</th>
<th>Value (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_{13})</td>
<td>30</td>
<td>(k_{12})</td>
<td>10</td>
</tr>
<tr>
<td>(k_{14})</td>
<td>25</td>
<td>(k_{13})</td>
<td>5</td>
</tr>
<tr>
<td>(k_{15})</td>
<td>20</td>
<td>(k_{25})</td>
<td>2</td>
</tr>
<tr>
<td>(k_{23})</td>
<td>25</td>
<td>(k_{24})</td>
<td>5</td>
</tr>
<tr>
<td>(k_{24})</td>
<td>20</td>
<td>(k_{25})</td>
<td>15</td>
</tr>
</tbody>
</table>

---

**Table 3.** Optimal terminating admittances for the analyzed numerical example

<table>
<thead>
<tr>
<th>(G_{L1}) (mS)</th>
<th>(L_{L1}) ((\mu)H)</th>
<th>(G_{L2}) (mS)</th>
<th>(L_{L2}) ((\mu)H)</th>
<th>(G_{L3}) (mS)</th>
<th>(L_{L3}) ((\mu)H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.4</td>
<td>524.0</td>
<td>23.0</td>
<td>443.0</td>
<td>22.6</td>
<td>370.0</td>
</tr>
</tbody>
</table>
Table 3 has been performed; SPICE returns an output power of 4.64 W, which corresponds to the value calculated by (10). Next, six simulations were performed by varying among the six parameters of interest one parameter at a time (either one of the three conductances $G_{Li}$ or one of the load inductors $L_{Li}$) while keeping all the others constant at their optimal value shown in the Table 3.

The achieved results are reported in Figures 5 and 6. They confirm the data provided by the theory for this example: a maximum power output is achieved when the loads are the ones calculated according to (9).

Note that maximizing the power output does not maximize the system efficiency $\eta$ (or power gain). SPICE returns an efficiency of 48.7%, which corresponds to the value calculated by (12). Moreover, the maximum power output configuration also does not realize a uniform power distribution. The relative output power delivered to the first, second, and third load is 59%, 29%, and only 12%, respectively.

5. Conclusions

A procedure to easily find the optimal load values to maximize the power transfer for a (reciprocal or nonreciprocal) multiport network has been presented. The proposed methodology is valid for any number of input and output ports. The calculation of the optimal loads requires just the knowledge of the admittance matrix of the system, which can be analytically calculated or measured. The reported formulas are validated through circuit simulations performed for a numerical example referring to a capacitive link using three transmitters and two loads.

6. References