Multi-dimensional representation of the electron density from satellite data

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Introduction

- In the last decade satellite missions have become a promising tool for measuring ionospheric parameters such as the electron density or the vertical total electron content (VTEC).

- Dual-frequency GPS observations can be used to determine the slant total electron content (STEC), i.e. the integral of the electron density along the ray-path of the signal.

- The insensitivity of ground-based observations to the radial geometry can be overcome by considering measurements from space-borne receivers flying on low-Earth-orbiting (LEO) satellites, e.g., the FORMOSAT-3/COSMIC 6-satellite constellation, CHAMP or GRACE.

- In the following, we present a general multi-dimensional approach for modeling spatio-temporal variations of the ionosphere.
Our multi-dimensional approach allows the following input strategies:

- **2D modeling**
  \[ \text{VTEC}(\lambda, \varphi) \]

- **3D modeling**
  \[ \text{VTEC}(\lambda, \varphi, t) \]
  \[ \text{N}(\lambda, \varphi, h) \]

- **4D modeling**
  \[ \text{N}(\lambda, \varphi, h, t) \]

Subtraction of the corresponding values from the reference model; here: International Reference Ionosphere (IRI-2000)

Residual input data (observations)
Area: global, regional, local

Parameterization:
B-spline-only expansions, combined expansions: B-splines/EOFs, trig. B-splines/B-splines, B-splines/Chapman function, SHs/EOFs, etc.

- Other ionospheric parameters like the maximum value of the electron density \( N_{mF2} \) can be modeled in the same manner.
The observation equation of the geometry-free linear combination reads

$$\phi_4(t) = \phi_1(t) - \phi_2(t)$$
$$= \alpha \cdot STEC(t) + \beta_R + \beta_S + \beta_{S,R} + e(t)$$

$\beta_R, \beta_S, \beta_{S,R}$ = inter-frequency delays in receiver $R$ and satellite $S$, ambiguity term ,
$\alpha, e(t)$ = constant, measurement error ,
$\phi_1(t), \phi_2(t)$ = phase observations on frequencies $f_1 = 1.5$ GHz and $f_2 = 1.2$ GHz ,
$STEC(t)$ = Slant Total Electron Content .

- The ambiguity term $\beta_{S,R}$ can be determined in a pre-processing step.
- The inter-frequency delays $\beta_R$ and $\beta_S$ can be determined in a common adjustment together with the parameters of the electron density approach we will present in the following.
STEC modeling

- The STEC is defined as the integral of the space- and time-dependent four-dimensional electron density $N(x, t)$ along the ray-path between the satellite $S$ and the receiver $R$, i.e.

$$\text{STEC}(t) = \int_{R}^{S} N(x, t) \, ds,$$

wherein

$$x = r \begin{bmatrix} \cos \varphi \cos \lambda, & \cos \varphi \sin \lambda, & \sin \varphi \end{bmatrix}^T = \text{geocentric position vector}$$

$$\lambda, \varphi, r = \text{longitude, latitude, radial distance}, \quad t = \text{time}$$

$$r = \rho + h, \quad \rho, h = \text{mean Earth radius, height}.$$

- We decompose the electron density into a given reference model $N_{\text{ref}}(x, t)$, computed from IRI, and an unknown correction term $\Delta N(x, t)$, i.e.

$$N(x, t) = N_{\text{ref}}(x, t) + \Delta N(x, t).$$
• Inserting Eq. (2) into Eq. (1) yields

\[ \text{STEC}(t) = \text{STEC}_{\text{ref}}(t) + \int_{R}^{S} \Delta N(x, t) \, ds. \]

• Approach: **3D B-splines**, i.e., a series expansion

\[
\Delta N(x, t) = \sum_{k_1=0}^{K_1-1} \sum_{k_2=0}^{K_2-1} \sum_{k_3=0}^{K_3-1} d_{k_1,k_2,k_3}(t) \phi_{k_1}^{J_1}(\lambda) \phi_{k_2}^{J_2}(\varphi) \phi_{k_3}^{J_3}(\theta)
\]

in 3D products of **1D scaling functions** \( \phi_{k}^{J}(\cdot) \) with \( J = \text{level (scale)} \) and \( k = \text{shift} \).
The substitution of the series expansion for $\Delta N(\omega, t)$ into the "STEC" equation yields

$$STEC(t) = STEC_{ref}(t) + \sum_{k_1=0}^{K_1-1} \sum_{k_2=0}^{K_2-1} \sum_{k_3=0}^{K_3-1} d_{k_1,k_2,k_3}(t) k_{k_1,k_2,k_3}(R,S),$$

wherein

$$k_{k_1,k_2,k_3}(R,S) = \int_{R}^{S} \frac{j_1}{j_1} \frac{j_2}{j_2} \frac{j_3}{j_3} dS,$$

The final observation equation for the geometry-free GPS observation $\phi_4(t)$ reads

$$\phi_4(t) - \beta_{S,R} - \alpha \cdot STEC_{ref}(t) = \alpha \cdot \sum_{k_1=0}^{K_1-1} \sum_{k_2=0}^{K_2-1} \sum_{k_3=0}^{K_3-1} d_{k_1,k_2,k_3}(t) k_{k_1,k_2,k_3}(R,S) + \beta_R + \beta_S + e(t),$$

unknown scaling coefficients

unknown delays
B-spline modeling

- As 1D scaling functions $\phi^J_k(x)$ of level $J$ we choose **normalized endpoint-interpolating quadratic B-spline functions** shown in the figure below for level $J = 3$. ($\# = 2^J + 2$)
B-spline modeling

- As 1D scaling functions $\phi^j_k(x)$ of level $J$ we choose **normalized endpoint-interpolating quadratic B-spline functions** shown in the figure below for level $J = 3$. ($# = 2^J + 2$)

- The figure on the left-hand side demonstrates the **localizing feature** of the 2D B-splines $\phi^j_{k_1}(x) \phi^j_{k_2}(y)$, since the higher the level value $J$ is chosen the sharper is the peak.

- Mathematically spoken B-splines are **compactly supported**. The „non-zero“ zone (influence zone, **foot print**) in the $(x,y)$-plane is similar to a **circle** or an **ellipse**.

- In the 3D case these non-zero zones are similar to **spheres** or **ellipsoids** (biaxial or triaxial).
• The centers \((k_1,k_2)\) of the foot prints of the 2D B-spline functions \(\phi_{k_1}^j(\lambda)\ \phi_{k_2}^j(\varphi)\) define a regular grid within the unit square.

• Due to their compact support (yellow foot prints) only B-splines related to the red dots, have non-zero entries in the observation equation shown before.

• The series (scaling) coefficient \(d_{k_1,k_2}\) can be determined by the GNSS observation related to the piercing point (green dot).
The centers \((k_1,k_2)\) of the foot prints of the 2D B-spline functions \(\phi_{k_1}^j(\lambda)\) \(\phi_{k_2}^j(\varphi)\) define a regular grid within the unit square.

Due to their compact support (yellow foot prints) only B-splines related to the red dots, have non-zero entries in the observation equation shown before.

The series (scaling) coefficient \(d_{k_1,k_2}\) can be determined by the GNSS observation related to the piercing point (green dot).

If the location of the observation lies close to the border of the foot print the coefficient is weakly determined → prior information for \(d_{k_1,k_2}\).
Procedure (adjustment, MRR, data compression)

Input data from GNSS, COSMIC, Altimetry, etc. and reference models

Prior information for the unknown scaling coefficients

Linear model with unknown variance components

Regularization step

Test for outliers

Estimation of the coefficients and the variance components

Test for significance

Estimated electron density, approximation on highest level

Wavelet decomposition of the electron density (MRR)

Data compression

Real-time processing
The relative GPS-LEO geometry is changing rapidly (including occultation measurements).
LEO measurements stabilize the estimation process.
Dual-frequency altimetry satellites measure always the vertical electron content (more or less only over the oceans).
Combination of input data

- For each of the **different observation types** (including altimetry and terrestrial measurements) it is possible to derive an **observation equation** for estimating the unknowns of the electron density B-spline model within a specific spatio-temporal (4D) region $\mathbb{B}^4$.

- We assume now that for each of the altogether $p$ techniques an **observation vector** $y_i$, $i = 1, \ldots, p$ is available, e.g. $y_1$ may contain all reduced GNSS geometry-free observations measured from ground stations within $\mathbb{B}^4$.

- Hence, for each observation vector $y_i$ a **Gauss-Markov model**

\[
y_i + e_i = A_i \beta \quad \text{with} \quad D(y_i) = \sigma^2_{y,i} P_{y,i}^{-1}
\]

- The quantities $\sigma^2_{y,i}$ and $P_{y,i}$ are the unknown variance factor and the given positive definite weight matrix.

- The vector $\beta$ consists of the unknown **B-spline scaling coefficients** $d_{k_1,k_2,k_3}(t)$, the unknown delays $\beta_R$ and $\beta_S$ and other auxiliary parameters. Note, the scaling coefficients have to be discretized w.r.t. the time.
With the prior information for the expectation vector \( E(\beta) = \mu_\beta \) and the covariance matrix \( D(\beta) = P_\beta^{-1} \) for the unknown B-spline scaling coefficients the additional linear model

\[
\mu_\beta + e_\beta = \beta \quad \text{with} \quad D(\mu_\beta) = \sigma^2_\beta P_\beta^{-1}
\]

can be formulated; \( e_\beta \) = error vector of the prior information, \( \sigma^2_\beta \) = unknown variance factor.

The combination of the models yields the normal equation system

\[
\left( \sum_{i=1}^{p} \frac{1}{\sigma^2_{y,i}} A_i^T P_{y,i} A_i + \frac{1}{\sigma^2_\beta} P_\beta \right) \hat{\beta} = \sum_{i=1}^{p} \frac{1}{\sigma^2_{y,i}} A_i^T P_{y,i} y_i + \frac{1}{\sigma^2_\beta} P_\beta \mu_\beta
\]

for the unknown parameter vector \( \beta \) with unknown variance components \( \sigma^2_{y,i}, \ i = 1, \ldots, p \) and \( \sigma^2_\beta \).

For solving this problem we apply a fast Monte-Carlo implementation of the iterative maximum-likelihood Variance Component Estimation (MCVCE, Koch and Kusche, 2001).
We chose 34 stations in Central and South America as well as 69 positions along the GPS satellite orbits. Hence, we evaluate altogether \(34 \times 69 = 2346\) simulated observations.

- The ray-paths with a zenith angle larger than 80° were neglected.
- For each measurement we have to evaluate the integrals over the 3-dimensional B-spline functions along the ray-paths. We perform this integration by **Gauss quadrature**.

For more details see:

Zeilhofer, C. et al. (2008): Regional 4-D modeling of the ionospheric electron density from satellite data and IRI. Advances in Space Research (JASR), accepted, in press

To test our B-spline approach we simulate GNSS observations. For that purpose we consider the *residual electron density*

\[ \Delta N(x,t) = N_{sim}(x,t) - N_{ref}(x,t) \]

wherein \( N_{sim}(x,t) \) are simulated electron density values and \( N_{ref}(x,t) \) are the values of the reference model, namely a low-pass filtered IRI version.

To simulate GPS observations we set time \( t = t_0 \) and integrate along the ray-paths over the residual electron density \( \Delta N(x,t) \) and get *residual STEC* values

\[ \Delta \text{STEC}(R,S) = \int_{R}^{S} \Delta N(x,t) \, ds \]

Replacing the integrand by the 3D series expansion for the residual electron density yields

\[ \Delta \text{STEC}(R,S) + e(R,S) = \sum_{k_1=0}^{K_1-1} \sum_{k_2=0}^{K_2-1} \sum_{k_3=0}^{K_3-1} d_{k_1,k_2,k_3} \, k_{k_1,k_2,k_3} (R,S). \]
The distribution of the data (receiver- and satellite-positions) influences the deviations.

We divide the area into 9 subareas and calculate the correlations and rms values of the deviations.

Over the continent we have high correlations and low rms values, this is due to the advantageous distribution of the ray-paths.

The two rows show respectively:

- **left panels**: 2 simulated residual VTEC data sets (bottom: higher frequency content in latitude)
- **mid panels**: estimated B-spline models of the levels (4,4,4) and (4,5,4) respectively
- **right panels**: deviations of the input data and the estimations
The distribution of the data (receiver- and satellite-positions) influences the deviations.

We divide the area into 9 subareas and calculate the correlations and rms values of the deviations.

The combination of **altimetry** and GNSS observations combines observations that lie mainly above the ocean and above the continent.
We add an additional linear model for altimetry observations. The observation equation reads

$$\Delta VTEC(R, S) + e(R, S) = \sum_{k_1=0}^{K_1-1} \sum_{k_2=0}^{K_2-1} \sum_{k_3=0}^{K_3-1} d_{k_1,k_2,k_3} k_{k_1,k_2,k_3} (R, S).$$

- The black crosses show the observation sites of the simulated altimetry measurements.
- The combination improves clearly the accuracy of the results over the oceans.
- The bottom panels show the combination results.
- The rms value decreases from 8.4 TECU to 6.9 TECU.
We add an additional linear model for altimetry observations. The observation equation reads

\[
\Delta VTEC(R, S) + e(R, S) = \sum_{k_1=0}^{K_1-1} \sum_{k_2=0}^{K_2-1} \sum_{k_3=0}^{K_3-1} d_{k_1,k_2,k_3} k_{k_1,k_2,k_3} (R, S).
\]
**COSMIC input data**

- **COSMIC/FORMOSAT-3** is the Constellation Observing System for Meteorology, Ionosphere and Climate and Taiwan's Formosa Satellite Mission #3, a joint Taiwan-U.S. project.

- The six COSMIC satellites were successfully launched on April 14th, 2006. Over the first year, the satellites had been gradually boosted from their initial orbit of 400 km to the final orbit of 800 km.

- Each spacecraft is equipped with three instruments, namely a GPS (occultation) receiver (GOX), a tiny ionospheric photometer (TIP) and a tri-band beacon (TBB).
In our investigation we calculate the electron density in a pre-processing step from so-called compensated STEC values by an "improved" Abel transform.

To be more specific, we do not assume locally spherical symmetry within the ionosphere, but consider the effect of large-scale horizontal gradients and/or inhomogeneous electron density distribution.

For more details see the paper: Tsai, Lung-Chih and Wei-Hsiung Tsai (2004): Improvement of GPS/MET ionospheric profiling and validation using the Chung-Li ionosonde measurements and the IRI model. (TAO,15(4), 589-607)

A "measured" VTEC value (observation) mean the sum of two parts:

- $VTEC_0$ is the integration of the calculated electron density along the vertical from bottom to orbital height.

- $VTEC_1$ is an extrapolated value provided by Lung-Chih Tsai using a Chapman layer model.
The table shows the statistical values of the COSMIC input data and the corresponding VTEC values from IRI.

The lower figure shows the “measured” COSMIC VTEC input data $VTEC_0$ between orbital height and the bottom of the ionosphere and the corresponding $VTEC_0$ values from IRI. (outlier detection was performed)

<table>
<thead>
<tr>
<th></th>
<th>COSMIC</th>
<th>IRI</th>
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</thead>
<tbody>
<tr>
<td>Mean</td>
<td>5.18</td>
<td>3.45</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3.22</td>
<td>2.03</td>
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<tr>
<td>RMS</td>
<td>6.10</td>
<td>4.00</td>
</tr>
<tr>
<td>Correlation coefficient</td>
<td>0.76</td>
<td></td>
</tr>
</tbody>
</table>
In case of **regional modeling** the ray-paths of **occultation** measurements might leave the region of investigation → **global model approach**.

The figure shows the **trigonometric B-spline** functions defined on the interval between 0° to 360°.

These functions are also **compactly supported**, i.e. the non-zero (influence) zone is finite.

In opposite to the endpoint-interpolating B-splines the trigonometric B-splines are “**wrapping around**”.

![Trigonometric B-spline](image)
We apply this global approach for modeling ΔVTEC, i.e. we introduce **trigonometric B-splines** in longitude (level 4).

For modeling the latitude- and time-dependency we again use **endpoint interpolating B-splines** (level 5 and level 3, resp.).

In the figures the crosses indicate the observation sites (blue and green colours mean negative values, orange positive values).

Structures are modelable only in regions where observations are available.
We apply this global approach for modeling $\Delta$VTEC, i.e. we introduce **trigonometric B-splines** in longitude (level 4).

For modeling the latitude- and time-dependency we again use **endpoint interpolating B-splines** (level 5 and level 3, resp.).

In the figures the crosses indicate the observation sites (blue and green colours mean negative values, orange positive values).

Structures are modelable only in regions where observations are available.
The panels show the estimated VTEC from COSMIC by adding the estimation of $\Delta$VTEC to the reference values from IRI.

The shown results are “snapshots” with a temporal resolution of 2 hours.

Estimated VTEC values can be calculated for each time $t$ of July 21st, 2006.

Validation, e.g. by ionosondes necessary.
Final remarks

- We demonstrated how a multi-dimensional model of ionospheric signals can be performed.
- We presented first a 3D B-spline approach with time-dependent coefficients for GNSS geometry-free observations.
- The accuracy was improved by incorporating altimetry observations.
- We further presented a 3D trigonometric/B-spline approach for modeling VTEC data from COSMIC.
- We showed just some components of the procedure. The next step will focus on the joint evaluation of the real data from GNSS, altimetry and COSMIC.
- Since the presented procedure can be applied to regional or local areas, features like the Equatorial Anomaly can be handled appropriately.
- Finally, an update of climatological parameters of IRI can be performed.