

A New Method to Estimate the Contribution of Geometrical Optics in Arbitrary Linear Stratified Planar Structures

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Abstract – Studying the contribution of geometrical optics in arbitrary linear stratified planar structures is of great importance in practical problems. In this article we propose a new procedure based on the Bresler–Marcuvitz transversalization method and equivalent network modeling that is useful for computing source contributions in Wiener–Hopf formulations of complex scattering problems where angular or stratified structures are present. The generalized Wiener–Hopf technique has demonstrated the capability to handle new complex canonical problems through both exact and semianalytical factorization methods.

1. Introduction

The Wiener–Hopf technique in its generalized form has been applied effectively in electromagnetic wave scattering problems for angular regions (wedge problems); see [1, 2] and references therein. Following the procedure first proposed in [3], we aim to extend the Wiener–Hopf technique in angular regions for arbitrary linear wave scattering problems [3–6]. This technique can be also extended to geometries containing angular regions or stratified planar regions; see, for instance, [7]. We start our formulation from electromagnetic applications [3–5] and extend the procedure to elasticity as reported in [6]. The method is based on two steps: the deduction of the generalized Wiener–Hopf equations for angular-region problems [3–6] and the solution of the equations using the semianalytical factorization procedure known as Fredholm factorization; see, for instance, [8, 9]. A key point in implementing the solution of generalized Wiener–Hopf equations for arbitrary linear stratified media via Fredholm factorization is the extraction of the source term that is related to geometrical-optics components. For this reason, this article is dedicated to estimating the contribution of geometrical optics in arbitrary linear stratified planar structures with the help of equivalent network models and the Bresler–Marcuvitz transversalization method.

2. Transverse Equations in Stratified Media

In this article we use only time harmonic fields with a time dependence specified by the factor $e^{j\omega t}$,

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which is omitted. In the absence of finite-located sources, Maxwell’s equations assume the abstract form [5, 10] in the Euclidean space of dimension 6:

$$(\Gamma_{\nabla} - W)\tilde{\Psi} = 0 \quad (1)$$

where

$$\tilde{\Psi} = |\tilde{\mathbf{E}}, \tilde{\mathbf{H}}|^t, \quad \Gamma_{\nabla} = \begin{vmatrix} 0 & \mathbf{1}_3 \times \nabla \\ \mathbf{1}_3 \times \nabla & 0 \end{vmatrix}, \quad (2)$$

$$W = j\omega \begin{vmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ -\boldsymbol{\zeta} & -\boldsymbol{\mu} \end{vmatrix}$$

with W containing the dyadic permittivity $\boldsymbol{\varepsilon}$, the dyadic permeability $\boldsymbol{\mu}$, and the additional coupling parameters $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ for general bianisotropic media.

We introduce Cartesian coordinates (z, x, y) and consider stratification along the y direction.

The study of the wave motion in stratified media is significantly simplified if we introduce the transverse equations of the fields. These equations involve only the components $\tilde{\Psi}_t$ of the field $\tilde{\Psi}$ that remain continuous along the stratification according to the boundary conditions on the interfaces. According to the boundary conditions, we have

$$\tilde{\Psi}_t = |\tilde{E}_z, \tilde{E}_x, 0, \tilde{H}_z, \tilde{H}_x, 0|^t = \mathbf{I}_t \cdot \tilde{\Psi} \quad (3)$$

where $\mathbf{I}_t = \text{diag}[1, 1, 0, 1, 1, 0]$. The transverse equations are obtained using [10] as reported in [5]:

$$-\frac{\partial}{\partial y} \tilde{\Psi}_t = \tilde{M} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right) \cdot \tilde{\Psi}_t \quad (4)$$

with matrix differential operator

$$\tilde{M} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right) = -\Gamma_y [(W_{yy} - \mathbf{I}_t \cdot \Gamma_{\nabla_t}) \cdot \hat{W}_y (\mathbf{I}_y \cdot \Gamma_{\nabla_t} - W_{yt}) + W_{tt}] \quad (5)$$

whose terms are explicitly defined and reported in [5].

One of the most important relations in the procedure is

$$\tilde{\Psi}_y = \hat{W}_y (\mathbf{I}_y \cdot \Gamma_{\nabla_t} - W_{yt}) \cdot \tilde{\Psi}_t \quad (6)$$

with $\tilde{\Psi}_y = |0, 0, \tilde{E}_y, 0, 0, \tilde{H}_y|^t$ that relates the discontinuous longitudinal component to transverse components without a partial derivative along y (only the third and the sixth rows are not null).

Note that the transverse equations (4) are defined in the Euclidean space of dimension 4 instead of 6, since the third and sixth rows are null for the definition (3).

In the following we consider invariance of the geometry along z as for stratified media. With this limitation, if the sources depend on an $e^{-j\alpha_0 z}$ factor, the total field also depends on the same factor—that is, $\tilde{\Psi}_c(y, \boldsymbol{\rho}) = \tilde{f}(y, x)e^{-j\alpha_0 z}$. Furthermore, to study the variation of the fields in x , we introduce the Fourier transform

$$f(y, \eta) = \int_{-\infty}^{\infty} \tilde{f}(y, x)e^{j\eta x} dx \quad (7)$$

That yields from (4) the ordinary differential equations

$$-\frac{d}{dy}\Psi_t(\eta, y) = M(\eta)\Psi_t(\eta, y) \quad (8)$$

with $M(\eta) = \tilde{M}(-j\alpha_0, -j\eta)$; see (5).

The definition of the explicit form of $M(\eta)$ and related properties need symbolic elaboration that can be performed with the help of software like Wolfram Mathematica [11]. In [5], $M(\eta)$ is reported for isotropic and anisotropic cases.

3. Circuit Model of an Arbitrary Semi-Infinite Layer

To introduce circuit modeling, we define as voltage and current the components in the Euclidean space of dimension 2 related to the Fourier transforms of $\tilde{\Psi}_t$. More precisely, we define

$$\Psi_t(\eta, y) = |\mathbf{V}(\eta, y), \mathbf{I}(\eta, y)|^t \quad (9)$$

with

$$\mathbf{V}(\eta, y) = \int_{-\infty}^{\infty} \begin{vmatrix} \tilde{E}_z(x, y) \\ \tilde{E}_x(x, y) \end{vmatrix} e^{j\eta x} dx, \quad (10)$$

$$\mathbf{I}(\eta, y) = \int_{-\infty}^{\infty} \begin{vmatrix} \tilde{H}_z(x, y) \\ \tilde{H}_x(x, y) \end{vmatrix} e^{j\eta x} dx$$

We consider the plane $y = y_0$ and an arbitrary stratified medium located at $y > y_0$ in the presence of arbitrary sources. The linearity of the problem imposes that $\mathbf{V}(\eta, y_0)$ and $\mathbf{I}(\eta, y_0)$ satisfy the equation

$$\mathbf{V} = \mathbf{V}_s + Z_e \mathbf{I} \quad (11)$$

where $\mathbf{V} = \mathbf{V}(\eta, y_0)$, $\mathbf{I} = \mathbf{I}(\eta, y_0)$, and $\mathbf{V}_s = \mathbf{V}_s(\eta)$ and $Z_e = Z_e(\eta)$ are called the Thevenin voltage and the Thevenin impedance, respectively. We estimate the Thevenin voltage \mathbf{V}_s at $y = y_0$ by imposing a perfect magnetically conducting boundary condition at $y = y_0$ —that is, $\mathbf{I} = 0$. The impedance Z_e is the 2×2 matrix that relates \mathbf{V} and \mathbf{I} when the sources are vanishing in the region $y > y_0$. The circuit model of the region $y > y_0$ is illustrated in Figure 1. The Norton representation is the

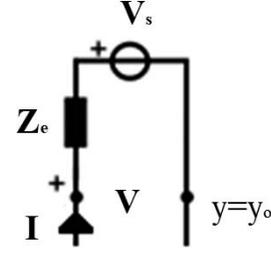


Figure 1. Thevenin representation of the half-space $y > y_0$.

dual circuit of the Thevenin one. Similar consideration apply in the stratified region located at $y < y_0$.

In the Thevenin model of a half-infinite layer, $Z_e = Z_e(\eta)$ represents the characteristic impedance of the medium. In particular, the matrix $\tilde{Z}_c = \tilde{Z}_c(\eta)$ is the characteristic impedance that relates \mathbf{V} and \mathbf{I} in the direction $y > y_0$ in the absence of sources, and the matrix $\tilde{Z}_c = \tilde{Z}_c(\eta)$ relates \mathbf{V} and $-\mathbf{I}$ in the direction $y < y_0$.

4. Circuit Model of an Arbitrary Multilayer Slab

Let us consider a homogeneous slab defined between $y = 0$ and $y = d$. The solution of (8) yields

$$\begin{vmatrix} \mathbf{V}_0 \\ \mathbf{I}_0 \end{vmatrix} = \mathcal{T} \begin{vmatrix} \mathbf{V}_d \\ \mathbf{I}_d \end{vmatrix}, \quad \mathbf{V}_0 = \mathbf{V}|_{y=0}, \quad \mathbf{I}_0 = \mathbf{I}|_{y=0}, \quad (12)$$

$$\mathbf{V}_d = \mathbf{V}|_{y=d}, \quad \mathbf{I}_d = \mathbf{I}|_{y=d}$$

with the transmission matrix of the slab $0 < y < d$ defined by

$$\mathcal{T} = \mathcal{T}(\eta) = e^{M(\eta)d} = \begin{vmatrix} \mathbf{A}(\eta) & \mathbf{B}(\eta) \\ \mathbf{C}(\eta) & \mathbf{D}(\eta) \end{vmatrix} \quad (13)$$

This 4×4 matrix has as elements the 2×2 matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} defined in terms of the M matrix of dimension 4 (see section 2). Figure 2 shows a convenient representation of (12) in terms of a two-port network model. This representation is also valid for a slab constituted by a cascade of s homogeneous consecutive slabs 1, 2, 3, ... In this case the transmission matrix of the multilayer slab is the product of the transmission matrices relevant to each slab:

$$\mathcal{T} = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \dots = e^{M_1 d_1} e^{M_2 d_2} e^{M_3 d_3} \dots \quad (14)$$

5. The Eigenvalues and the Eigenvectors

The eigenvalues and eigenvectors of the matrix M reported in (8) are very important in studying the solution of the equation. For example, they allow the evaluation functions of M such as the exponentials that appear in (13) and (14). We study the eigenvalues and eigenvectors of M defined in the Euclidean space of dimension 4 (see section 2):

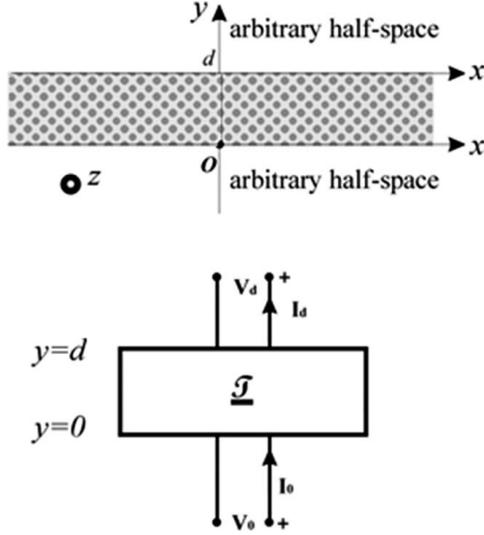


Figure 2. Top: Slab filled by an arbitrary linear medium. Bottom: Network equivalent model.

$$M(\eta)\Psi^{(i)}(\eta) = \gamma_i(\eta)\Psi^{(i)}(\eta) \quad (15)$$

where $\gamma_i(\eta)$ and $\Psi^{(i)}(\eta)$ are, respectively, the eigenvalues and the eigenvectors. Since M is a semisimple matrix, we have

$$M = UJU^{-1} \quad (16)$$

where $J = \text{diag}[\gamma_1(\eta), \dots, \gamma_4(\eta)]$, $U = [\Psi^{(1)}(\eta), \dots, \Psi^{(4)}(\eta)]$, and $\Psi^{(i)}(\eta) = [\psi_1^{(i)}(\eta), \psi_2^{(i)}(\eta), \psi_3^{(i)}(\eta), \psi_4^{(i)}(\eta)]^t$.

In the presence of a passive medium, we observe that two eigenvalues (say, γ_1, γ_2) present a nonnegative real part and the other two, γ_3, γ_4 , present a nonpositive real part, whence $\Psi^{(i)}(\eta)$ ($i=1,2$) are called progressive eigenvectors and $\Psi^{(i)}(\eta)$ ($i=3,4$) are called regressive eigenvectors. The eigenvectors of M provide the characteristic impedance of a medium. In fact [9], we have

$$\vec{Z}_c = \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix} \cdot \begin{vmatrix} \psi_3^{(1)} & \psi_3^{(2)} \\ \psi_4^{(1)} & \psi_4^{(2)} \end{vmatrix}^{-1}, \quad (17)$$

$$\vec{Z}_c = \begin{vmatrix} \psi_1^{(3)} & \psi_1^{(4)} \\ \psi_2^{(3)} & \psi_2^{(4)} \end{vmatrix} \cdot \begin{vmatrix} \psi_3^{(3)} & \psi_3^{(4)} \\ \psi_4^{(3)} & \psi_4^{(4)} \end{vmatrix}^{-1}$$

Furthermore, the transmission matrix $e^{M(\eta)d}$ (13) is obtained by $e^{M(\eta)d} = Ue^{Jd}U^{-1}$, where $e^{Jd} = \text{diag}[e^{\gamma_1(\eta)d}, \dots, e^{\gamma_4(\eta)d}]$.

6. Plane Waves in an Arbitrary Medium

Plane waves in an arbitrary medium are solutions of the transverse equations (4) with (5) of the form

$$\Psi_{\mathbf{t}} = \Psi_o e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (18)$$

where $\mathbf{r} = [z, x, y]^t$ is the observation point, $\mathbf{k} = [k_y, \alpha_o, \eta_o]^t$ is the propagation vector, and Ψ_o is a

constant vector of dimension 4. Taking into account that $\frac{\partial}{\partial y} = -jk_y$, (4) becomes

$$jk_y\Psi_o = M(\eta_o)\Psi_o \quad (19)$$

where k_y are related to the eigenvalues of matrix M as defined in section 5. For a given set of α_o, η_o , we have four possible propagation constants k_y and four polarizations Ψ_o :

$$jk_{yi} = \gamma_i(\eta), \quad \Psi_{oi} = \Psi^{(i)}(\eta), \quad i = 1, 2, 3, 4 \quad (20)$$

We call the plane waves where $i=1, 2$ progressive and the plane waves where $i=3, 4$ regressive. Examples of values are given in [12]. For each value of k_y (we omit the subscript i) we define a propagation vector and a propagation constant,

$$k = |\mathbf{k}| = \sqrt{\alpha_o^2 + \eta_o^2 + k_y^2}, \quad \tau_o = \sqrt{k^2 - \alpha_o^2} \quad (21)$$

that identify the direction of the wave in terms of zenithal angle β and azimuthal angle φ_o : $\eta_o = -\tau_o \sin \varphi_o$, $\alpha_o = k \cos \beta$. We recall that the field $\Psi_{\mathbf{t}}$ contains only the transverse components of the electromagnetic field. The discontinuous components E_y and H_y are related to the transverse ones through (6).

6. The Reflection Problem

Considering the network modeling of a semi-infinite medium, we build the network representation of the reflection problem as shown in Figure 3.

In the network representation, the impedances Z_{c1} and Z_{c2} are the characteristic ones obtained in (17). We observe that the model is complete, considering sources in region 1 ($y > 0$) that yield (through imposing perfect magnetically conducting termination at $y=0$) the Thevenin voltage $2\mathbf{V}^{\text{inc}}(\alpha_o, \eta_o)$, where $\mathbf{V}^{\text{inc}}(\alpha_o, \eta_o)$ is the incident voltage at $y=0$. The incident voltage $\mathbf{V}^{\text{inc}}(\alpha_o, \eta_o)$ is related to the regressive incident plane wave that constitutes the source in region 1. Without loss of generality, we suppose the presence of only one kind of incident plane: $\mathbf{V}^{\text{inc}}(\alpha_o, \eta_o) = \mathbf{V}^{\text{inc}} = V_o \mathbf{V}^{(i)}$, where V_o is the intensity of the plane wave and $\mathbf{V}^{(i)}$ is the voltage part (that is, the first two components) of one of the regressive eigenvectors $\psi_c^{(i)}$ ($i=3,4$) of the matrix M relevant to medium 1—that is, $\mathbf{V}^{(i)} = [\psi_1^{(i)}, \psi_2^{(i)}]^t$. Analyzing the circuit model of Figure 3, we have

$$\mathbf{I} = -(Z_{c1} + Z_{c2})^{-1} 2\mathbf{V}^{\text{inc}}$$

$$\mathbf{V} = -Z_{c2}\mathbf{I} = T\mathbf{V}^{\text{inc}} = \mathbf{V}^{\text{inc}} + \Gamma\mathbf{V}^{\text{inc}} = \mathbf{V}^{\text{inc}} + \mathbf{V}^{\text{ref}} \quad (23)$$

where $\mathbf{V}^{\text{ref}}(0) = \mathbf{V}^{\text{ref}} = \Gamma\mathbf{V}^{\text{inc}} = \Gamma\mathbf{V}^{\text{inc}}(0)$ defines the reflected voltage at $y=0$ and the 2×2 reflection matrix Γ is given by

$$\Gamma = T - 1_3 = 2Z_{c2}(Z_{c1} + Z_{c2})^{-1}$$

$$= (Z_{c2} + Z_{c1})(Z_{c1} + Z_{c2})^{-1} \quad (24)$$

When the structure is excited or illuminated with one

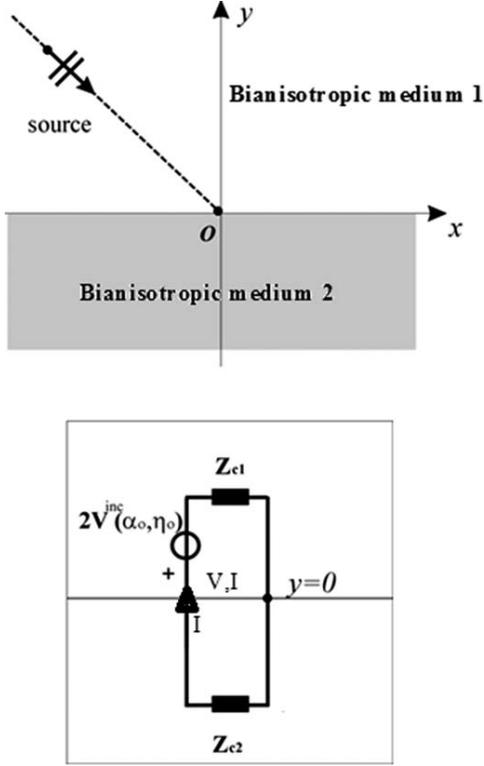


Figure 3. Top: Plane wave reflection between two semi-infinite homogenous bianisotropic media. Bottom: Network modeling based on the Thevenin's equivalence.

regressive wave (either $i = 3$ or $i = 4$), in general we get as reflected waves all the progressive reflected waves ($i = 1, 2$) of medium 1 and all the regressive transmitted waves ($i = 3, 4$) of medium 2. We can evaluate the coupling coefficient in terms of the reflection matrix Γ . Taking into account that the reflected wave

$$\mathbf{V}^{\text{ref}}(y) = C_{o1}e^{-\gamma_1 y}\mathbf{V}^{(1)} + C_{o2}e^{-\gamma_2 y}\mathbf{V}^{(2)} \quad (25)$$

contains all the progressive plane waves present at $y > 0$, we obtain the excitation coefficients C_{o1}, C_{o2} by considering that at $y = 0$ we get

$$\begin{aligned} \mathbf{V}^{\text{ref}}(0) &= \Gamma \mathbf{V}^{\text{inc}}(0) = C_{o1}\mathbf{V}^{(1)} + C_{o2}\mathbf{V}^{(2)} \\ &= \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix} \begin{vmatrix} C_{o1} \\ C_{o2} \end{vmatrix} \end{aligned} \quad (25)$$

which yields

$$\begin{aligned} \begin{vmatrix} C_{o1} \\ C_{o2} \end{vmatrix} &= \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix}^{-1} \mathbf{V}^{\text{ref}}(0) \\ &= \begin{vmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{vmatrix}^{-1} \Gamma \mathbf{V}^{\text{inc}}(0) \end{aligned} \quad (26)$$

Similar considerations apply for the evaluation of plane waves transmitted in medium 2 ($y < 0$). By

indicating with γ_{ii} and $\mathbf{V}_i^{(i)}$ ($i = 3, 4$) the regressive eigenvalues and eigenvectors of the matrix $\mathbf{M}_i(\eta_o)$ defined in medium 2, for $y < 0$ we have

$$\mathbf{V}(y) = C_{to3}e^{-\gamma_{t3}y}\mathbf{V}_t^{(3)} + C_{to4}e^{-\gamma_{t4}y}\mathbf{V}_t^{(4)} \quad (27)$$

$$\begin{vmatrix} C_{to3} \\ C_{to4} \end{vmatrix} = \begin{vmatrix} \psi_{t1}^{(3)} & \psi_{t1}^{(4)} \\ \psi_{t2}^{(3)} & \psi_{t2}^{(4)} \end{vmatrix}^{-1} T\mathbf{V}^{\text{inc}}(0) \quad (28)$$

7. Conclusions

This work proposes a new effective method to estimate the contributions of geometrical optics in arbitrary linear stratified planar structures, based on equivalent network models and Bresler–Marcuvitz transversalization theory. The method is particularly useful for starting the analysis of novel complex canonical problems constituted of angular or stratified structures with the generalized Wiener–Hopf technique, as in [13].

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