New Insight in the Propagation of Wave Fronts in the Vicinity of a Refractive Object

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Abstract – From Maxwell's equations in Riemannian geometry, the geodetic ray path is defined as the path where the electromagnetic energy efficiently travels through the medium. To validate that the electromagnetic wave fronts are determined by the geodetic lines, wave fronts are rigorously computed from an integral equation formulation in the specified case of a dielectric sphere.

1. Introduction

In our analysis, we start with Maxwell's equations in tensor format going from a Cartesian geometry to a Riemannian geometry with a given metric tensor. From the behavior of the stationarity of the Poynting vector, we conclude that the metric tensor is functionally dependent on the refractive index of the medium.

Our first choice is a simple diagonal metric tensor in which the diagonal terms are given by the inverse square of the refraction index. This represents an orthogonal transformation and leads to the standard ray theory. However, it cannot accommodate the structure of the ray in vacuum in the vicinity of the boundary of an object, because the metric tensor only has an expression in material media and has a Dirac form at the interface with vacuum.

We extend our analysis to a nonorthogonal transformation. To guarantee uniqueness, we borrow the trace of the operator from our first choice, the diagonal operator. This leads to a new symmetric metric tensor as a function of a refractional potential for all media in vacuum. The refractional potential gives rise to a *tension* that determines the definition of the geodetic line. This tension is also present in a vacuum outside the object. The geodetic lines are determined by Fermat's principle of least time.

2. Scaled Maxwell's Equations

We consider electromagnetic wave propagation with complex time factor $\exp(-\mathrm{i}\omega t)$, where i is an imaginary unit, ω is the radial frequency, and t is the time. In a source-free domain, with Cartesian coordinates $\mathbf{x} \in R^3$, we write Maxwell's equations in the frequency domain as

$$\nabla \times \mathbf{H} + \mathrm{i}\omega\varepsilon\mathbf{E} = 0 \tag{1}$$

$$\nabla \times \mathbf{E} - \mathrm{i}\omega\mu\mathbf{H} = 0 \tag{2}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{x}, \omega)$ is the electric field vector, $\mathbf{H} = \mathbf{H}(\mathbf{x}, \omega)$ is the magnetic field vector, $\varepsilon = \varepsilon(\mathbf{x}, \omega)$ is the electric permittivity, and $\mu = \mu(\mathbf{x}, \omega)$ is the magnetic permeability. We neglect absorption so that all material parameters are real valued.

Next, we introduce the scaled electromagnetic field vectors as

$$\tilde{\mathbf{E}} = \mathbf{E}/\sqrt{Z}\,, \qquad \tilde{\mathbf{H}} = \sqrt{Z}\,\mathbf{H} \tag{3}$$

in which $Z = \sqrt{\mu/\varepsilon}$ is the wave impedance.

Using this scaling in Maxwell's equations, we obtain

$$\frac{\sqrt{Z}}{n}\nabla \times (\tilde{\mathbf{H}}/\sqrt{Z}) + \frac{\mathrm{i}\omega}{c_0}\tilde{\mathbf{E}} = 0 \tag{4}$$

$$\frac{1}{n\sqrt{Z}}\nabla \times (\sqrt{Z}\tilde{\mathbf{E}}) - \frac{\mathrm{i}\omega}{c_0}\tilde{\mathbf{H}} = 0 \tag{5}$$

where the refractive index $n = n(\mathbf{x}, \omega)$ follows from

$$n = \sqrt{(\varepsilon\,\mu)\,/\,(\varepsilon_0\mu_0)} = c_0/c \tag{6}$$

in which $c = 1/\sqrt{\varepsilon\,\mu}$ and $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ are the electromagnetic wave velocities in a material medium and in a vacuum. Note that (4) and (5) represent the Maxwell equations for the scaled electromagnetic field vectors in a background medium with permittivity ε_0 and permeability μ_0.

At this point, we may not assume that the waves in this background medium travel with the wave velocity c_0. The curl operators in Maxwell's equations are replaced by the medium-dependent curl operators, which determine the spatial dependency of the electromagnetic wave field. Let us denote the fastest path of the wave as the geodetic line. Although, with the help of present-day computer codes, a more or less complete solution of Maxwell's equations is possible, the structure of the geodetic lines is difficult to observe from the numerical solution. We, therefore, investigate the nature of Maxwell's equations in a different coordinate system.

3. Maxwell's Equations in Tensor Notation

We introduce a Riemannian geometry with position vector $\bar{x}^j$ and symmetric metric tensor and its conjugate as, for example [1],

$$g_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \bar{x}^j}\,, \quad g^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^k}\,, \quad \text{and } g_{ik}g^{kj} = \delta_i^j \tag{7}$$

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Note that the Einstein summation convention with repeated indices is used.

We introduce the electric field vector $\bar{E}_i$, the magnetic field vector $\bar{H}_i$, and the partial derivative $\bar{\partial}_i$ in the Riemannian geometry as $\{\bar{E}_i, \bar{H}_i, \bar{\partial}_i\} = \frac{\partial x^j}{\partial \bar{x}^i}\{\tilde{E}_j, \tilde{H}_j, \partial_j\}$.

In tensor notation, the scaled Maxwell's equations are written as the covariant equations

$$\frac{g_{ij}\sqrt{Z}}{n\sqrt{g}} e_{jkl}\bar{\partial}_k(\bar{H}_l/\sqrt{Z}) + \frac{\mathrm{i}\omega}{c_0}\,\bar{E}_i = 0 \tag{8}$$

$$\frac{g_{ij}}{n\sqrt{g}\sqrt{Z}} e_{jkl}\bar{\partial}_k(\sqrt{Z}\,\bar{E}_l) - \frac{\mathrm{i}\omega}{c_0}\,\bar{H}_i = 0 \tag{9}$$

The permutation tensor e_{ijk} denotes the Levi–Civita symbol. It is obvious that any solution of the Maxwell equations in a Riemannian geometry needs the specification of the refraction index and the impedance in whole space. Both material parameters occur in the curl operators. To investigate the energy transport in the Riemannian space in more detail, we introduce the complex Poynting vector $\bar{S}^k$ as

$$\bar{S}^k = (1/\sqrt{g})\, e_{kjl}\, \bar{E}_j\bar{H}_l^* \tag{10}$$

where g is the determinant of the metric tensor g_{ij} and the asterisk denotes complex conjugate. Next, we contract the complex conjugate of (8) with $\bar{E}^i$ and (9) with $\bar{H}^{i*}$.

Adding the two results and combining various terms, we obtain

$$\frac{1}{n\sqrt{g}}\bar{\partial}_k(\sqrt{g}\,\bar{S}^k) + \mathrm{i}\omega\varepsilon_0|\bar{\mathbf{E}}|^2 - \mathrm{i}\omega\mu_0|\bar{\mathbf{H}}|^2 = 0 \tag{11}$$

Taking the real part of (11) and multiplying the result with $n\sqrt{g}$, we arrive at

$$\bar{\partial}_k[\sqrt{g}\,\mathrm{Re}(\bar{S}^k)] = 0 \tag{12}$$

which represents the conservation law of energy transport in the Riemannian geometry.

Within the accuracy of geometrical optics, the curvature of the ray is determined by the refractive index only, see [2, p. 115]. We choose the metric tensor g_{ij} to be a function of the refraction index n.

4. Orthogonal Transformation

Choosing for the simple orthogonal transformation

$$\frac{\partial x^j}{\partial \bar{x}^i} = \frac{1}{n}\delta^j_i\,, \quad \frac{\partial \bar{x}^i}{\partial x^j} = n\,\delta^i_j \tag{13}$$

yields the diagonal metric tensor

$$g_{ij} = \frac{1}{n^2}\,\delta_{ij} \tag{14}$$

This choice of transformation just scales the local spatial behavior of the refraction index. It does not take into account the global refraction dependency.

Later in this article, we show that the geodetic line evaluation on the basis of this metric leads to the well-known ray theory. In a vacuum domain outside an object $\mathbb{S}$, it leads to propagation along a straight line in the Cartesian space. Inside the object $\mathbb{S}$, this transformation leads to curved paths according to the standard ray theory on the basis of a high-frequency approximation of the wave field.

5. Nonorthogonal Transformation

Let us assume that g is a twice differentiable function not equal to one inside $\mathbb{S}$ and then $g \equiv 1$ does not hold at all points in vacuum. This implies that g is a harmonic function, determined by its values at the interface of the object. In other words, the presence of object $\mathbb{S}$ changes the direction of energy transport in each finite domain, because $g \not\equiv 1$.

Physically, it means that a wave approaching the object $\mathbb{S}$ is *feeling* the object before it has reached it. It will propagate along a path where the variation of $\sqrt{g}$ $\mathrm{Re}(\bar{S}^k)$ is stationary. The direction will change at locations where $\sqrt{g} \not\equiv 1$. In this way, the object shows its emergence to the incident wave field, see also Feynman [3, chapter 26.5].

To develop a transformation and hence a geodetic formulation that accounts for the global dependency of the refraction index, we proceed as follows. The divergence of $\bar{x}^k(\mathbf{x})$ (trace of the operator) is taken from the contraction of the second relation of (13) as

$$\frac{\partial \bar{x}^k}{\partial x^k} = 3n \tag{15}$$

Subsequently, we introduce the difference vector between the spatial points $\bar{x}^k$ and x^k as

$$f^k(\mathbf{x}) = \bar{x}^k(\mathbf{x}) - x^k \tag{16}$$

in which f^k is a continuous and differentiable vector field.

Next, taking the divergence of f^k and using (15), we obtain

$$\mathrm{div}\,\mathbf{f} \;=\; \frac{\partial f^k}{\partial x^k} \;=\; \frac{\partial \bar{x}^k}{\partial x^k} - 3 \;=\; 3(n-1) \tag{17}$$

We further require that the transformation matrix be symmetric so that $\partial \bar{x}^i/\partial x^j = \partial \bar{x}^j/\partial x^i$ or, in other words, that the tension $\mathbf{f}$ be curl free.

After a differentiation of (16) with respect to x^k, it follows that the new nonorthogonal transformation is given by

$$\frac{\partial \bar{x}^k}{\partial x^i} = \delta^k_i + \frac{\partial f^k}{\partial x^i} \tag{18}$$

Using the property that the tension $\mathbf{f}$ is curl free, together with the expression for the divergence of $\mathbf{f}$, the Helmholtz decomposition theorem for a curl-free vector provides us with the unique expression; also see (17):

$$\mathbf{f} = -\mathrm{grad}\,\Phi\,, \qquad \Phi(\mathbf{x}) = \int_{\mathbf{x}'\in\mathbb{S}} \frac{3\,[n(\mathbf{x}')-1]}{4\pi|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}V \quad (19)$$

Obviously, f^k is the tension due to the difference in refractive index with respect to vacuum. We define Φ as the refractional potential, and we denote f^k as the refractional tension.

This representation is valid under the condition that $n-1$ vanishes at the boundary surface of the domain $\mathbb{S}$. Thus, (18) and (19) define our transformation matrix. They hold for any distribution of the refractive index inside domain $\mathbb{S}$.

Note that the expression of the refractive potential yields a nonzero value outside $\mathbb{S}$. This confirms that the refractive index distribution inside the object $\mathbb{S}$ not only determines the spatial coordinate transformation inside this object but also outside. Hence, the geodetic lines in the vacuum domain around $\mathbb{S}$ are influenced by the inner refractive index of the object.

Substitution of the expression for the tension $\mathbf{f}$ into (18) yields the transformation matrix as

$$\frac{\partial \bar{x}^k}{\partial x^i} = \delta_i^k - \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\int_{\mathbf{x}'\in\mathbb{S}} \frac{3\,[n(\mathbf{x}')-1]}{4\pi|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}V \qquad (20)$$

To compare this transformation matrix with the one of (13), we analyze the domain integral on the right-hand side of (20) in more detail. When $\mathbf{x}\in\mathbb{S}$, the evaluation of the domain integral has to be interpreted as the Cauchy principal value.

To this end, we consider the contribution of the integration over a spherical domain $\mathbb{S}_\delta$ with vanishing radius δ and center point $\mathbf{x}$. Using $\lim_{\delta\downarrow 0}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\int_{\mathbf{x}'\in\mathbb{S}_\delta}\frac{3\,[n(\mathbf{x})-1]}{4\pi|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}V = -[n(\mathbf{x})-1]\,\delta_i^k$, we observe that (20) becomes

$$\frac{\partial \bar{x}^k}{\partial x^i} = n(\mathbf{x})\,\delta_i^k + \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}\oint_{\mathbf{x}'\in\mathbb{S}} \frac{3\,[n(\mathbf{x})-n(\mathbf{x}')]}{4\pi|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}V \quad (21)$$

Here, $\oint$ denotes that the integral has to be interpreted as its Cauchy principal value.

If we neglect the integral completely, the transformation matrix becomes identical to the one of (13). We remark that on the right-hand side of (21), the first term represents the local and orthogonal part of the transformation, while the second term stands for the global part.

6. Construction of the Geodetic Line

In the construction of the geodetic line, we consider the scalar arc length $\overline{\mathrm{d}s}$ along the curved geodetic line, given by

$$\overline{\mathrm{d}s} = \left[\overline{\mathrm{d}x}^k\,\overline{\mathrm{d}x}^k\right]^{\frac{1}{2}} = \left[\frac{\partial \bar{x}^k}{\partial x^l}\frac{\partial \bar{x}^k}{\partial x^j}\mathrm{d}x^l\mathrm{d}x^j\right]^{\frac{1}{2}} \qquad (22)$$

At this point, we switch to the matrix representation of the tensors and introduce the curvature matrix C_{ij} as a representation of the symmetric transformation tensor $\partial\bar{x}^j/\partial x^i$ of (18), namely,

$$\overline{\mathrm{d}s} = \left[C_{kl}\,C_{kj}\,\mathrm{d}x^l\mathrm{d}x^j\right]^{\frac{1}{2}}, \qquad C_{ij} = \delta_{ij} + \partial_i f_j \qquad (23)$$

Because the matrix C_{jk} is real and symmetric, an eigenvalue decomposition with positive eigenvalues exists and the sum of the eigenvalues is equal to the trace.

By inspection of (18), we learn that in the transform geometry, the coordinate axes are spanned by the components of the tension $\mathbf{f}$. The latter is directed in the normal direction to the surface $\Phi =$ constant.

This normal $\boldsymbol{\nu}$ is an eigenvector belonging to the matrix C_{ij}, because

$$C_{ij}\nu_j = \lambda_\nu \nu_i\,, \quad \lambda_\nu = 1 + \nu_j\partial_j|\mathbf{f}|\,, \quad \nu_i = f_i/|\mathbf{f}| \qquad (24)$$

In our additional analysis, we choose $\boldsymbol{\nu}$ as the principal unit vector of the transformation, complemented with the other two eigenvectors τ_1 and τ_2 with corresponding eigenvalues λ_{τ_1} and λ_{τ_2}.

Because the trace of a symmetric operator is equal to the sum of the eigenvalues, from (15), it follows that $\lambda_\nu + \lambda_{\tau_1} + \lambda_{\tau_2} = 3n$. Next, we consider the optical length, see [2, p. 115], of the left equation of (23). Using the eigenvalue decomposition, we may write the optical length as $\overline{\mathrm{d}s} = n^{\mathrm{g}}(\mathbf{x},\hat{\mathbf{s}})\,\mathrm{d}s$, where $n^{\mathrm{g}}(\mathbf{x},\hat{\mathbf{s}}) = \left[\lambda_{\tau_1}^2\hat{s}_{\tau_1}^2 + \lambda_{\tau_2}^2\hat{s}_{\tau_2}^2 + \lambda_\nu^2\,\hat{s}_\nu^2\right]^{\frac{1}{2}}$ and $\hat{\mathbf{s}}$ is the unit vector with components $\hat{s}_{\tau_1} = \mathrm{d}x_{\tau_1}/\mathrm{d}s$, $\hat{s}_{\tau_2} = \mathrm{d}x_{\tau_2}/\mathrm{d}s$, and $\hat{s}_\nu = \mathrm{d}x_\nu/\mathrm{d}s$.

The virtual refractive index $n^{\mathrm{g}}(\mathbf{x},\hat{\mathbf{s}})$ controls the path of the geodetic line in a similar way as the refractive index $n(\mathbf{x})$ controls the path of optical rays. Note that the virtual refractive index is not only determined by the local position of the geodetic line, but it also depends on the direction of the geodetic line at this position. We construct this geodetic line by considering the classic differential equation for the evolution of an optical ray path, see [2, pp. 121-122], but we replace the physical refractive index n by the virtual counterpart n^{g}, namely,

$$\frac{d[n^{\mathrm{g}}(\mathbf{x},\hat{\mathbf{s}})\,\hat{s}_j]}{ds} = \partial_j\,n^{\mathrm{g}}\,, \quad \text{with } \hat{s}_j = \frac{dx_j}{ds} \qquad (25)$$

where $x_j = x_j(s)$ is the trajectory of the geodetic line, s is the parametric distance along this trajectory, and $\hat{s}_j$ is the tangential unit vector along the geodetic line.

Note that this differential equation applies to refractive indices, which are invariant with respect to the direction of the geodetic path. However, the explicit Euler integration of this differential equation updates the ray position and ray direction so that only the previous information of position and direction about the associated path segment is used.

7. Numerical Example

We consider ray propagation from an electric dipole in the vicinity of a sphere with origin at $\mathbf{x} = (0,0,0)$, a radius of 60 m, and a refractive index of 1.5. The dipole is located at $\mathbf{x} = (-120,0,0)$ m and oriented in the horizontal x_1 direction. The x_2-axis is in

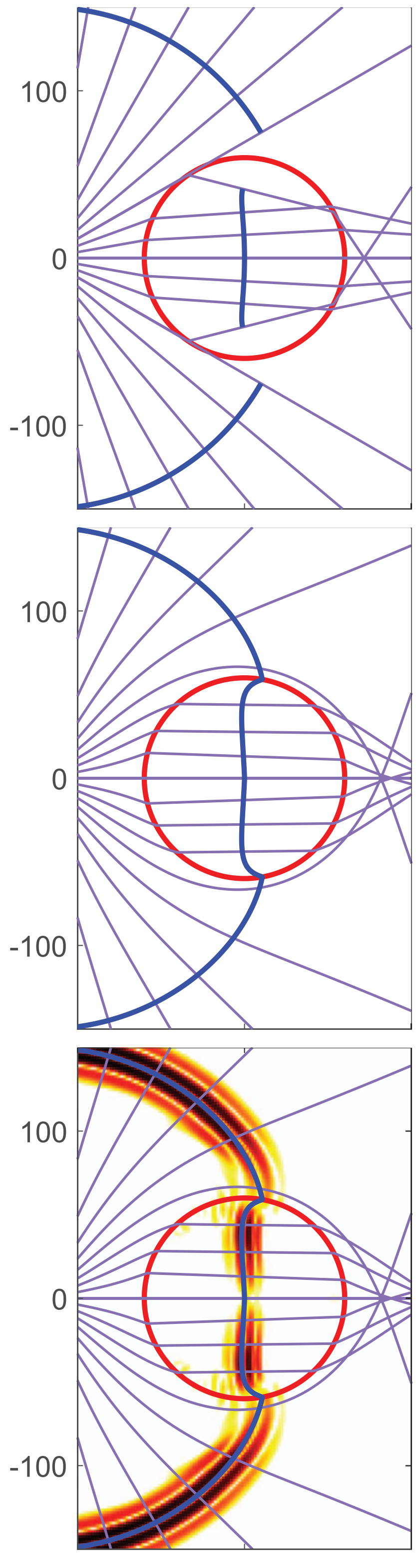

Figure 1. Ray paths (purple lines) and its wave front at $t = 0.5\ \mu$s (thick blue curves; top); Geodesics and geodetic wave front in the same configuration (middle); similar to the middle picture but now with the actual wave front as overlay (bottom).

the vertical direction. Comparing the results for standard ray theory with the geodesics shows clearly the difference (see top and middle panels of Figure 1).

The standard ray has a definition inside and outside the object, which are not connected. The geodesic wave front feels the object and shows how inside and outside are connected. To verify our results, we carry out a full three-dimensional numerical simulation on the basis of the contrast-source integral equation [4] for the electric field vector in the frequency domain. The electric dipole moment is described by a Gaussian function with negligible values outside the frequency range of 0 MHz to 40 MHz. The resulting electric field values are transformed back to the time domain, using a 128-point fast Fourier transform routine. These experiments (see bottom panel of Figure 1) corroborate our wave-front analysis that only the geodesics support the wave front of the numerical simulation.

8. Conclusions

This result has consequences for the bending of electromagnetic waves around the sun. Apart of coronal effects, Einstein's conclusion [5] is that the bending of light passing a massive star, the sun, is caused by the heaviness of the light in reaction to the gravitational force of the sun. We are not debating this conclusion, but we see, on the basis of Maxwell's equation, that there is room, next to the gravitational tension, for an additional refractional tension. For interstellar communication [6], it may add a significant correction to the gravitational lensing.

9. References

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