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Hilbert Hotel: An experimental realization in polarization topological vortex beam

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Abstract

Hilbert hotel paradox or infinite hotel paradox is a thought experiment proposed by David Hilbert in 1921 for visualizing the concept of infinity. The paradox describes an imaginary hotel with an infinite number of single occupancy rooms occupied by an infinite number of guests, later popularised as Hilbert hotel. In the paradox, if a new guest appears in the "fully" occupied Hilbert hotel, the manager can always find a room for the new guest by simply shifting the old guest to the adjacent room and making the first room vacant. Recently, there have been numerous attempts happened among the optics community to visualize this paradox in the optical beams utilizing the dynamics of phase singularities near the half-integer vortex beams. In this work, we are describing our efforts to visualize the paradox experimentally on an optical vector beam using polarization singularities. We are also proposing an experimental way to generate and control the fractional vortex beams from a single spiral phase plate and a supercontinuum source. This work will be a good example of the usefulness of structured beams in visualizing complex mathematical concepts. The work will also be useful for fundamental and applied research based on fractional vector beams.

1 Introduction

Hilbert's Hotel, a subject of much interest in countably infinite sets in mathematics originally ascribe to David Hilbert in 1924 but later on popularized by George Gamow assumes a hotel having an infinite number of rooms marked with all natural numbers 1,2,3, ... initially full with visitors [1]. He pointed out that it is always possible to accommodate a new visitor if each guest in the room shifts to the next highest room. Likewise, any number of vacancies say N can be in the hotel by shifting them to N rooms up. This process leads to N number of vacant rooms and can be described by the mathematical formula $\infty + N \to \infty$. Thereafter, any number of vacant rooms can be created, and thus it is said that the Hilbert hotel is simultaneously completely occupied and infinitely vacant.

Over the past decades, efforts have been made in order to observe such strange behavior in reality in many different ways in classical and quantum systems. Oi *et al.* introduced a cavity quantum electrodynamics platform where all the quantum amplitudes are shifted to the next level leaving an unoccupied vacuum state [2]. In classical optics, phase singularities in optical wavefronts have turned out to be a great platform to emulate the Hilbert hotel paradox [3]. An optical vortex beam may possess orbital angular momentum having an azimuthal phase variation of 0 to $2\pi l$ resulting in a dark core at the center of the beam where l is an integer and defines the topological charge or order of the vortex beam [4,5]. l is a conserved quantity under perturbation and thus in a closed ensemble optical vortices can be created and annihilated in pairs. This discreteness of the vortex beam enables us to manifest the Hilbert hotel paradox [6]. Recently, Chen *et. al.* experimentally demonstrated [7] a simple interference setup to observe the Hilbert hotel paradox in a fractional vortex beam as proposed in Ref. [3]. Like the optical vortices, the fractional vortex beams are also realized through the holographic technique by illuminating a spatial light modulator (SLM) containing a phase mask corresponding to the desired fractional vortex by Gaussian beam. However, the change in the fractional vortex order requires a change in the phase mask and a small tweaking of the experiment. Despite the flexibilities of SLMs in terms of dynamic phase modulation, and wide wavelength coverage, the generation of high-power vortex beams are mostly using spiral phase plates (SPPs). The SPPs are the elements having helically increasing optical thickness. Based on the design wavelength, λ, and the vortex order, l, the step height, h is engineered in such a way that the incident Gaussian beam acquires azimuthal shift of $2\pi l$. The vortex order of the beam can be defined as

$$l = (n_\lambda - l)h/\lambda \tag{1}$$

where n_λ is the wavelength-dependent refractive index of the material. As a result, the SPPs are very wavelength specific and can not be used to generate integer vortex of order l for laser wavelength away from the designed wavelength, λ. However, from Eq. (1) it is clearly evident that one can continuously vary the vortex order from l to $2l$ by varying the beam wavelength from λ to $\lambda/2$ with a modification factor due to the variation of refractive index with wavelength. Therefore, by simply changing the wavelength of the Gaussian beam one can generate vortex beams of varying fractional charge in ease. Recently, Wang *et. al.* theoretically realized that the beams possessing nonuniform polarisation across their transverse cross-section known as vector beams can also manifest the behavior of countably infinite sets while passing through a fractional nonuniform polarization element [8]. Here, we report, for the first time to the best of our knowledge, on the simple experimental demonstration of the Hilbert hotel in a finite-size fractional vector beam. Using a double-ramp SPP having transverse thickness variation corresponding to the vortex order $l = 2$ at the designed wavelength of 1064 nm in a modified Mach-Zehnder interferometer and supercontinuum laser tunable across 400-800 nm, we have generated a vector vortex beam with tunable fractional topological order and verified the Hilbert hotel.

2 Experimental setup

The schematic of the experimental setup for the realization of the Hilbert hotel paradox is shown in Figure 1. A supercontinuum laser (NKT Photonics) with an average power of 2 W and tunable across 400 nm to 2200 nm is used as the primary laser for the experiment. Using a variable tunable filter, we tuned the laser wavelength in the visible range across 400 nm to 800 nm with a minimum laser bandwidth of 10 nm. The coherence length of the laser is calculated to be $\sim 50\ \mu$m at 700 nm. The use of a beam expander comprised of pair of plano-convex lenses L1 and L2 of focal lengths $f1 = 50$ mm and $f2 = 300$ mm, respectively, results in a collimated Gaussian beam of diameter (full width at half maximum) of 6 mm. The combination of a half-wave plate (HWP1), and polarizing beam splitter cube (PBS1) are used to control the laser power in the experiment. On the other hand, the HWP2 controls the laser power in the modified Mach–Zehnder interferometer (MZI) comprised of PBS2-3, and mirrors, M1-6. We used a delay stage in one of the arms in order to temporally overlap the two beams at the output of the MZI. A doubly-ramp SPP made of fused silica having a topological charge of $l = 2$ at the designed wavelength of 1064 nm was kept in one arm of the MZI to generate a vector beam. The electric field at the output of the MZI can be written as

$$E_1 = a\left|E^H, l\right\rangle + b\left|E^V, 0\right\rangle. \qquad (2)$$

The final electric field after the quarter wave-plate 1 (QWP1) can be written as

$$E_2 = a\left|E^{LCP}, l\right\rangle + b\left|E^{RCP}, 0\right\rangle. \qquad (3)$$

here, LCP and RCP are left- and right circular polarization of light, a and b are the relative weightage of the two orthogonally polarized beams. QWP2, HWP3, and PBS4 are used for polarization Stoke's measurements of the vector beam.

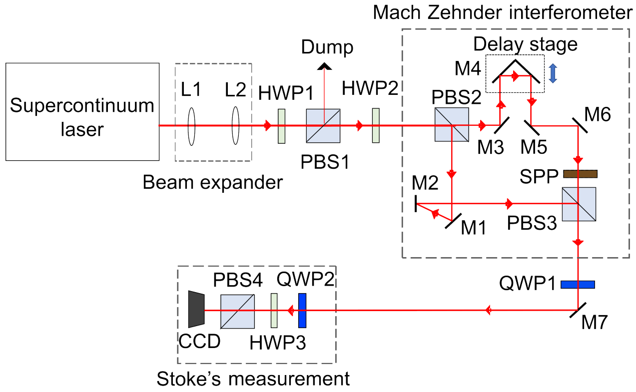

Figure 1. Schematic of the experimental setup: L1-2, plano convex lenses; HWP1-HWP3, $\lambda/2$ plates; PBS1-4, polarizing beam splitter cubes; M1-7, dielectric mirrors; SPP, spiral phase plate of order 2; QWP1-2, $\lambda/4$ plates; CCD, charged coupled devices camera.

3 Results and discussion

To understand the variation of the topological charge of the SPP of order l=2 with wavelength, we have plotted Eq. (1) considering the refractive index variation of fused glass with wavelength with the results shown in Figure 2. As evident from Figure (2), the topological charge varies from l=2 at 1064 nm to l= 4 at 544 nm instead of 532 nm due to the variation of the refractive index of fused silica with wavelength. However, the net topological charge, l of the double-ramped SPP is 3 at λ = 712 nm. Experimentally we observed a doughnut-shaped intensity profile at both 1064 nm and 544 nm with increased dark core size, and the truncated intensity profile as shown in the inset of Figure 2 at 712 nm confirming the fractional vortex beam. Since the wavelength of the supercontinuum source is tuned by a filter, the variation of the topological charge in the current experiment is very easy and straightforward [9]. Hence, by simply tuning the pump wavelength of the supercontinuum

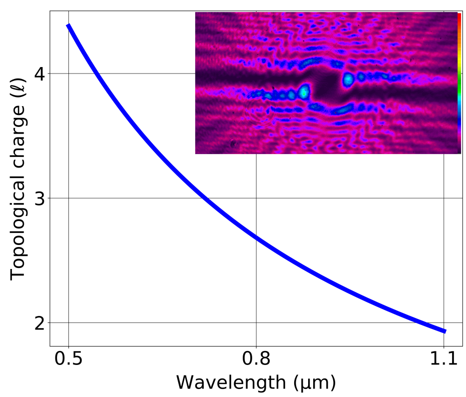

Figure 2. Variation of Topological charge of the SPP with the pump wavelength. The near field intensity profile of the beam clearly shows two singular lines on either side of the center for λ = 712 nm (l = 3.000) (inset).

laser, we can also generate a vector beam, with the electric field equation given in Eq. (3), of continuously tunable fractional topological charge.

In order to observe the Hilbert hotel behavior in fractional polarization singular beams, we measured the polarization characteristics of the beam while changing the laser wavelength from 760 nm to 658 nm with the results shown in Figure 3. As evident from the first column, (a-e), Figure 3, the ellipse orientation distribution of the vector beam for λ = 760 nm corresponding to theoretical l value of 2.8 showing two lemon singularities identified by “+" at the center of the beam. However, in the case of λ = 738 nm (l = 2.88), we clearly see the appearance of two lemon and star (denoted by “-") pairs along the polarization singular lines in addition to the two lemons at the beam center. Further change in the laser wavelength to λ = 712 nm (l = 3.0), we could observe an infinite series of lemon and star pairs along both the singular lines in addition to the former lemons at the beam center. Due to the finite beam size, the infinite number of

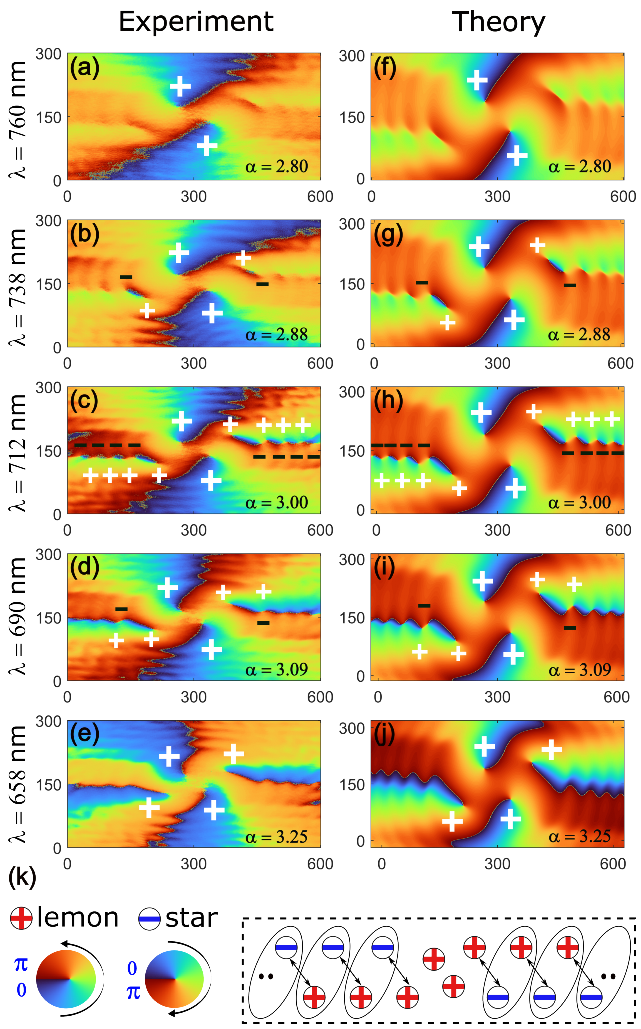

Figure 3. Illustration of the evolution of polarization singularities while changing the pump beam wavelength from 760 nm to 658 nm, experiment (a-e), theory (f-j); Orientation of “lemon" and “star" singularities denoted by “+" and “-" respectively and pictorial view of their evolution (k).

pairs is restricted only to four pairs which can be very large with the expansion of the beam size. This situation can be assumed to be equivalent to the Hilbert hotel model with a countably infinite number of “rooms" identified by lemon singularity, “+", fully occupied by the “guest" represented by the star singularity, “-". Hence, one can make a one-to-one correspondence between rooms and guests as $\infty \to \infty$. As we decrease the laser wavelength λ = 690 nm corresponding to l = 3.09, we clearly observe the annihilation of lemon and star singularities from the most distant point of the beam. The annihilation process of the lemon and star singularities can be understood as follows. Let us consider the right singularity line of the beam. As we increase l from 3.00 to 3.09, in order to accommodate extra lemon singularity into the system, the star singularity in the 2^{nd} pair will move to its right, connected and then annihilate with the lemon singularity of the 3^{rd} pair and subsequently the N^{th} star singularity will annihilate with the $(N+1)^{th}$ lemon singularity. The same is observed in the left singular line of the beam except that the star singularities will move to its left and annihilate with the lemon singularities in the next pair. At λ = 658 nm (l = 3.25) (fig. 3e), we clearly see the star singularity in the 1^{st} pair annihilates with the lemon in the 2^{nd} pair in both the singular lines and this is how the system produces two extra lemons in order to conserve the net topological charge of the vortex. This process of accommodating new lemons is similar to the Hilbert hotel model where each guest (here star singularity) moved to the next highest room (here lemon singularity) in order to accommodate new guests in the hotel. Our theoretical simulation (f-j) for all five wavelengths is in good agreement with the experimental results Therefore, the system acts as an $\infty \to \infty + 2$ version of Hilbert hotel.

4 Conclusion

In this contribution, we have designed a simple experimental scheme to verify the Hilbert hotel model in fractional polarization singular beams. In contrast to the previous experiments, our system enables us to produce fractional vector beams by use of a supercontinuum laser and an SPP. The generic experimental scheme could help us understand the behavior of complex polarization-sensitive optical elements, and thus can be useful in many fields including designing novel devices, quantum communication, and sensing.

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