Foster's Reactance Theorem for a Multiport, and its Application to Q Factor Measurement

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Abstract

The classical Foster's reactance theorem relates the input impedance or admittance to the stored energy inside a lossless one-port network. Such relation has been successfully applied to the measurement of Q factor for a resonator given its one-port response. Sometimes, however, a resonator is naturally monitored at more-than-one ports. Although there are various methods for Q measurement given two-port network parameters, it remains open as for whether the Foster-based method can be extended to such scenario. In this paper, we first derive the Foster's theorem for the multiport case, providing the relation between the stored energy and the *N*-port **Z** or **Y** matrix. Based on the result, we then propose a new method for Q-factor estimation. A feature of this method is that we directly use the whole **Z** or **Y** matrices to estimate Q, without reducing it to a one- or two-port network. Two simulation examples are provided to demonstrate the feasibility as well as the detailed operation of this method.

1 Introduction

Accurate measurement of the quality (Q) factor of a resonator is critical to many applications such as material characterization. There has been a vast amount of methods for Q measurement documented in the literature [1], of which one important category [2-3] determines the Q based on frequency-domain network responses such as the S, Y, and Z parameters as they are readily available from VNA measurement. In particular, the method of [4] applies the Foster's reactance theorem [5] to estimate the internally stored energy of a resonator given its one-port Z or Y parameters, which constitutes the numerator in the definition of Q. This method does not assume any lumped equivalent circuits for the resonator, thus having the advantages of being capable of measuring multiple closely-spaced resonances as well as very-low-Q resonators. One example in [4] demonstrates accurate estimation of Q as low as 1.57.

Sometimes, however, a resonator is naturally monitored at multiple ports. An example is the reverberation chamber with several antennas connected to the outside. Although there are various methods for Q measurement given two-port responses like [6], the aforementioned Foster-based method [4] does not seem to have been extended to such scenario. In particular, we are not aware of any literature that describes the Foster's theorem for a multiport network.

One may argue that given an *N*-port response, we can always reduce it to a one- or two-port and then apply the existing methods. However, the termination schemes that we adopted in the reduction process will certainly affect the resonance. Intuitively, if we terminate a port by a resistive load, the Q will reduce; if we terminate a port by a short or open, the resonance frequency may change. Therefore, if a resonator is originally constructed with multiple ports, it should be better to estimate the Q based directly on the multiport data.

In this paper, we will first derive the Foster's theorem for the multiport case, based on which we then propose a new method for Q measurement. Two examples are provided to illustrate the feasibility and operation of the method.

2 Foster's Theorem for a Multiport

The classical Foster's theorem [5] states that for a lossless one-port with input impedance $Z = jX$ and admittance $Y = 1/Z = jB$, the energy stored in the network is given by

$$W = \frac{1}{4}|I|^2\frac{dX}{d\omega} = \frac{1}{4}|V|^2\frac{dB}{d\omega} \tag{1}$$

The derivation can be found in [5]. In particular, we note that the following relation is obtained during the proof:

$$V\frac{\partial I^*}{\partial\omega} + \frac{\partial V^*}{\partial\omega}I = -4jW \tag{2}$$

where V and I are the port voltage and current, respectively. Now, if the same procedure as [5] are followed for an N-port network, we will find us arriving at

$$\sum_{i=1}^{N} V_i\frac{\partial I_i^*}{\partial\omega} + \frac{\partial V_i^*}{\partial\omega}I_i = -4jW \tag{3a}$$

or, written in matrix notation,

$$\dot{\mathbf{I}}^{\mathrm{H}}\mathbf{V} + \dot{\mathbf{V}}^{\mathrm{H}}\mathbf{I} = -4jW \tag{3b}$$

where the superscript H denotes conjugate transpose and the overdot the derivative with respect to ω. For a lossless multiport, the **Z** and **Y** matrices are purely imaginary [7]. Substituting $\mathbf{V} = j\mathbf{X}\mathbf{I}$ or $\mathbf{I} = j\mathbf{B}\mathbf{V}$ into (3b), we have

$$W = \frac{1}{4}\mathbf{I}^{\mathrm{H}}\dot{\mathbf{X}}\mathbf{I} = \frac{1}{4}\mathbf{V}^{\mathrm{H}}\dot{\mathbf{B}}\mathbf{V} \tag{4}$$

which is the multiport generalization of (1). One important thing to note from (4) is that the stored energy depends not only on the magnitude of the port voltages or currents as in (1), but also on the *distribution* (or, pattern) of voltages or currents across the N ports.

3 Measurement of Q

The definition of Q is

$$\mathrm{Q} = \omega_0 \frac{W}{P_{\mathrm{loss}}} \tag{5}$$

where ω_0 and P_{loss} are the resonance frequency and power loss, respectively. The numerator, W, is the energy stored inside the resonator, and is computed in [4] using (1). Specifically, we first measure broadband S parameters, and then convert to either $Z = R + jX$ or $Y = G + jB$. Resonance is identified to be the frequency where $X = 0$ or $B = 0$, while the slope of X or B is computed from the data to get W by (1). The power loss is given by

$$P_{\mathrm{loss}} = \frac{1}{2}|I|^2 R = \frac{1}{2}|V|^2 G \tag{6}$$

Combining (1) and (6), we get

$$\mathrm{Q} = \frac{\omega_0}{2R}\frac{dX}{d\omega} = \frac{\omega_0}{2G}\frac{dB}{d\omega} \tag{7}$$

The expression (7) is the core of the method in [4]. One may notice that the Foster's theorem (1) is proved for a lossless network, while for a real resonator, it is always lossy. The validity of using (1) to estimate W for a resonator thus becomes questionable.

Although we are not aware of any rigorous analysis for the accuracy of applying (1) on a lossy network, it appears that one of the two equalities in (1) remains approximately valid. Specifically, if the resonator is series-RLC-like, then the expression in terms of X, i.e., the first equality in (1), is a good estimate for W. For parallel-RLC resonance, likewise, the expression in terms of B provides a good estimate for W. Intuitively, for a series-RLC resonator, the presence of a nonzero R does not alter the slope of X greatly, while for parallel-RLC, the presence of G does not alter the slope of B. Therefore, in practice we first need to identify the *type* of the resonance, and then apply the right expression.

Now, extending to the multiport case, we write P_{loss} as

$$P_{\mathrm{loss}} = \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I} = \frac{1}{2}\mathbf{V}^{\mathrm{H}}\mathbf{G}\mathbf{V} \tag{8}$$

Note that (8) only accounts for loss *internal* to the network; losses due to loadings at the ports are excluded. Combining (8) with (4), we obtain the following estimates for Q:

$$\mathrm{Q} = \omega_0 \frac{\mathbf{I}^{\mathrm{H}}\dot{\mathbf{X}}\mathbf{I}}{2\mathbf{I}^{\mathrm{H}}\mathbf{R}\mathbf{I}} = \omega_0 \frac{\mathbf{V}^{\mathrm{H}}\dot{\mathbf{B}}\mathbf{V}}{2\mathbf{V}^{\mathrm{H}}\mathbf{G}\mathbf{V}} \tag{9}$$

This is the main equation of the proposed method. One apparent difference of (9) from (7) is that the expression now depends on the **V** or **I** vector. Different excitation schemes may result in different values. Then, what *exactly* is the quality factor of the resonator?

We recall the meaning of resonance: the energy stored in the system is constantly changing (oscillating) between two different forms (e.g., E and H fields), and the averaged energy is the same in these two forms. Therefore, at resonance frequency $X = 0$ or $B = 0$ for a one-port, because the external does not need to provide any imaginary power to the network.

With the same line of reasoning, the resonance of an N-port network should be at frequencies where det(**X**) = 0 or det(**B**) = 0, and that the port currents and voltages satisfy **XI** = 0 or **BV** = 0, i.e., **I** or **V** is in the *null space* of the matrix **X** or **B**, respectively.

We thus have a procedure for Q measurement: first, the plots of det(**X**) and det(**B**) are scanned to find out the frequencies of resonance; next, we calculate the null-space vectors for **X** and **B** at those frequencies; finally, knowing the **I** and **V** vectors, we then compute Q by (9).

4 Examples

Below, we use two examples to validate the proposed method, as well as to illustrate the detailed operations. The first example is a transmission line resonator as shown in Fig. 1, with length 180° at 1 GHz. The two ports are at the two sides of the microstrip. The S parameters are solved by Ansoft Designer SV [8], and exported into Python for post-processing.

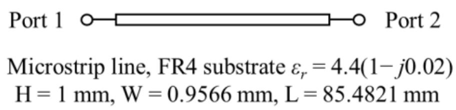

Figure 1. A microstrip $\lambda/2$ resonator.

After converting the S parameters to Z and Y parameters, we plot the determinants of **X** and **B** as shown in Fig. 2. It can be seen that det(**X**) = 0 and det(**B**) = 0 at 1 GHz, which reveals the resonance frequency.

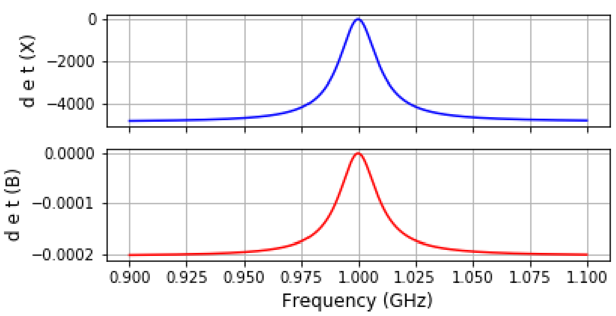

Figure 2. The det(**X**) and det(**B**) of the resonator in Fig. 1.

Knowing the resonance frequency, we then extract the null-space vectors of **X** and **B** at 1 GHz. Something tricky, however, must be noted here. We first plot the X_{11}, X_{21}, B_{11}, and B_{21} around 1 GHz as shown in Fig. 3. Due to symmetry and reciprocity, the other two X/B parameters are omitted.

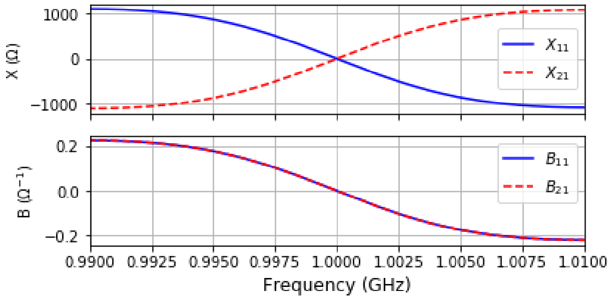

Figure 3. The X_{11}, X_{21}, B_{11}, and B_{21} around 1 GHz.

It can be seen that, for this resonator, **X** approaches the zero matrix at 1 GHz, and so is **B**. If we indeed take them as zero matrices, then every vector would be a null-space vector of **X** and **B**. Is that correct?

If we look at the **X** and **B** *very close* to 1 GHz but not exactly, we will find that the **X** and **B** have the following *asymptotic* forms:

$$\mathbf{X} \to x_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (10a)$$

$$\mathbf{B} \to b_0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (10b)$$

where x_0 and b_0 are constants that tend to 0 at 1 GHz. The forms of (10) can be readily observed from Fig. 3. Now, the null-space vectors of (10) are clear:

$$\mathbf{I} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (11)$$

Note that these two vectors are consistent with the physical phenomenon: for a $\lambda/2$ resonator, the currents at the two ports are in-phase if the ports are shorted; similarly, the voltages at the two ports are out-of-phase if the ports are left opened. We see that (11) in fact *defines* the conditions of resonance: for a $\lambda/2$-long transmission line to actually be a resonator, its two ports must be either both shorted or both opened.

Next, the derivatives of **X** and **B** at 1 GHz are computed using finite difference, given by (12). We must be careful not to mistake them as having the same form as (10); the derivatives of X_{11} and X_{21} (B_{11} and B_{21}) are slightly different, though not observable in the scale of Fig 3.

$$\dot{\mathbf{X}} = \begin{bmatrix} -3.3984 & 3.4002 \\ 3.4002 & -3.3984 \end{bmatrix} \times 10^{-5} \quad (12a)$$

$$\dot{\mathbf{B}} = \begin{bmatrix} -6.9319 & -6.9355 \\ -6.9355 & -6.9319 \end{bmatrix} \times 10^{-9} \quad (12b)$$

The **R** and **G** matrices are

$$\mathbf{R} = \begin{bmatrix} 2180.2 & -2179 \\ -2179 & 2180.2 \end{bmatrix} \quad (13a)$$

$$\mathbf{G} = \begin{bmatrix} 0.44487 & 0.44464 \\ 0.44464 & 0.44487 \end{bmatrix} \quad (13b)$$

Using (9), we obtain the Q values: 48.982 from **X** and **R**, and 49.001 from **B** and **G**. The theoretic Q factor of a $\lambda/2$ resonator is $\beta/2\alpha$ [7], where α and β are the attenuation and phase constants, respectively. Using α = 3.3 dB/m and β = 36.73 rad/m provided by Designer, we obtain Q = 48.3, which is in good agreement with our estimates.

One may notice that, for this example, the required accuracy for the Z and Y parameters is very high (5 digits). In practical VNA measurements, this level of accuracy is almost impossible to achieve. The root of the problem, as we will see, is the locations of the ports. It is known in circuit theory that, for a *lossless* $\lambda/2$ transmission line, the **Z** and **Y** matrices simply do not exist! Therefore, it can be appreciated that when a small amount of loss is introduced, the Z and Y parameters will be highly sensitive at the resonance frequency.

The second example is shown in Fig. 4, which is the same $\lambda/2$ resonator as Fig. 1 but observed at different locations. Specifically, the port-2 is at 1/3 distance from the right end which is left opened. The total length ($L_1 + L_2$) remains the same (85.4821 mm), so the resonance is at 1 GHz as before.

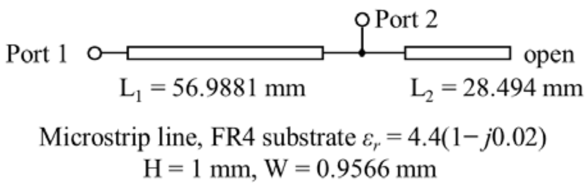

Figure 4. The same microstrip $\lambda/2$ resonator as Fig. 1 but with different port location.

Fig. 5 shows the determinant plots, from which we see that det(**X**) and det(**B**) are *still* zero at 1 GHz. This is worth pointing out, because it verifies that det(**X**) = 0 and det(**B**) = 0 are indeed the conditions of resonance.

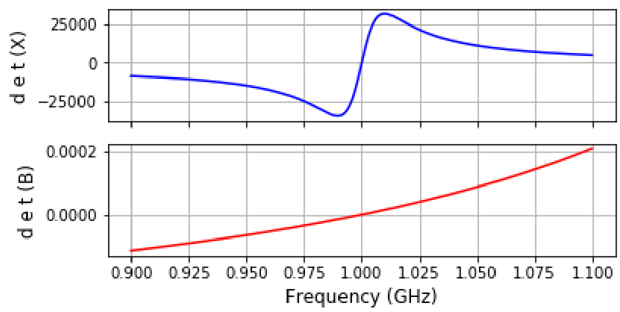

Figure 5. The det(**X**) and det(**B**) of the resonator in Fig. 4.

The plot of X_{11}, X_{21}, X_{22}, B_{11}, B_{21}, and B_{22} around 1 GHz are shown in Fig. 6. Now, because the circuit is not symmetric, $X_{11} \neq X_{22}$ and $B_{11} \neq B_{22}$.

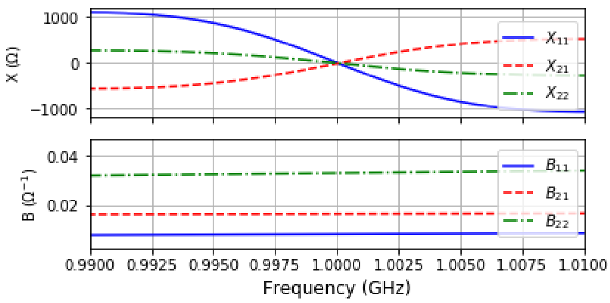

Figure 6. The X_{11}, X_{21}, X_{22}, B_{11}, B_{21}, and B_{22} around 1 GHz.

In this example, **X** again tends to the zero matrix at resonance. The same method could be used: finding the asymptotic form of **X** near 1 GHz, and then extract the null-space vector. However, since **B** is much more well-behaved at 1 GHz, below we only consider using **B** to estimate Q. At 1 GHz, the **B**, $\dot{\mathbf{B}}$, and **G** are given by

$$\mathbf{B} = \begin{bmatrix} 0.0082464 & 0.016490 \\ 0.016490 & 0.032983 \end{bmatrix} \tag{14a}$$

$$\dot{\mathbf{B}} = \begin{bmatrix} 6.3461 & 3.1735 \\ 3.1735 & 15.874 \end{bmatrix} \times 10^{-12} \tag{14b}$$

$$\mathbf{G} = \begin{bmatrix} 0.00040787 & 0.00020396 \\ 0.00020396 & 0.0010197 \end{bmatrix} \tag{14c}$$

We take the singular vector of **B** that corresponds to the smallest singular value as the null-space vector, which is

$$\mathbf{V} = \begin{bmatrix} -0.89443 \\ 0.44720 \end{bmatrix} \tag{15}$$

Using the above values and the second equality in (9), we obtain the estimated Q = 48.9, which is consistent with the theoretic value 48.3. Also, the accuracy requirement for this example is much relaxed. Specifically, if we only keep two significant digits in (14), we would get an estimate of Q = 48.5, which is even closer to the theoretic value.

The null-space vector (15) is worth noting. It is known that if the line is lossless and open at both ends, then the voltage distribution along the line is $V(x) = V_0\cos(\beta x)$ (let port-1 be at $x = 0$). As port-2 is at 2/3 distance from port 1, or $x = \lambda/3$, the voltage at port-2 is $V_0\cos(2\pi/3)$ or $-V_0/2$. Therefore, the null-space vector (15) in fact coincides with the voltage pattern on the *unloaded* resonator. This is a remarkable consistency.

5 Conclusions

In this paper, we derived the multiport version of Foster's reactance theorem, and applied it to the measurement of Q factor. The distinct feature of this method is that we could use the whole multiport data directly, instead of reducing it to a one- or two-port network. During the procedure, we need to solve the null-space vector of the **X** or **B** matrix at resonance, which turns out to correlate with the voltage or current patterns on the unloaded resonator. Two simulation examples were provided to demonstrate the feasibility of this approach, though the advantages compared to other existing methods were not fully understood. Future works include testing the method using three- or four-port data, as well as real measurements. To reveal the benefits of using Foster's theorem, we may also consider low-Q resonators, as well as multiple closely-spaced resonances.

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