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Algebraic Topological Method: An Alternative Modelling Tool for Electromagnetics

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Abstract

We begin with a simple question: what is the real need for field vectors and differential equations while modelling electromagnetic problems? Particularly, when all what we can practically measure in electromagnetics are only scalars, the traditional approach to modelling in electromagnetics is proving to be more mathematical than physical. In addition, the use of differential equations takes us through an indirect path for modelling underlying physics. We cannot directly translate continuous differential equations into numerical algorithms because computers need discrete formulations. In this work, we discuss a direct discrete and computationally competitive tool using only physically measurable scalar quantities called the algebraic topological method for modelling different electromagnetic problems. We will also highlight areas of current and future research in this domain.

1 Introduction

Maxwell-Heaviside equations are traditionally modelled using the differential or integral formulation. Over the years, various computational methods were developed employing different strategies for spatial and temporal discretisations [1–13]. In this paper, we are presenting a radically different *non-mainstream* tool called the *algebraic topological method* - ATM [14–22]. Unlike many traditional methods, in ATM we use only physically measurable scalar quantities *avoiding the use of differential equations* and *field vectors*. Using the mathematical tools of algebraic topology, we directly get discrete formulations for the underlying electromagnetic phenomena. The physically measurable quantities such as potential, current, electric & magnetic fluxes, and charge content are defined as *cochains* on *topological objects* such as points, lines, surfaces, and volumes. The connection between physical quantities and their respective topological objects is at the core of direct discrete ATM formulation for modelling electromagnetic phenomena [23, 24]. It is important to note that the framework of ATM is more general and goes beyond the application to electrodynamics. We can employ the basic ATM tools to model also other phenomena, for example, thermoelectric, thermodynamics, etc. We will briefly discuss the building blocks of ATM method in the following sections and present the final ATM formulations used to model electromagnetic problems.

2 Chains & cochains

Consider a star-shaped domain shown in Fig. 1. For simplicity, we have discretized this domain using tetrahedral cells. Each tetrahedral cell (volume) is called a 3-simplex, where the number 3 denotes the dimension of the simplex. Likewise, we have 2-, 1-, and 0-simplexes in the domain representing surfaces (triangular faces), lines, and points, respectively. A collection of the simplexes is called as *chains*. A collection of topological objects is called 0-, 1-, 2-, or 3-chain when it represents a set of points, lines, surfaces, or volumes, respectively as shown in Fig. 1. Mixing of topological objects of different dimensions is not allowed in the ATM framework. That is, a k-chain has strictly only collection of k-simplexes.

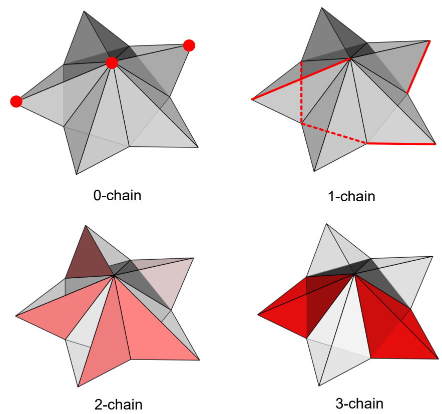

Figure 1. Example: 0-, 1-, 2-, and 3-chains. The respective cochains are potentials, electromotances, fluxes, and charge-contents defined on these chains.

3 Boundary & coboundary operators

The power and elegance of ATM lie in two inter-related tools, namely *boundary* and *coboundary* operators [25]. Let us first explain the boundary operator. The boundary operator is a mathematical tool, which operates on the underlying topological object, which could be lines, surfaces, or volumes. Note that there is no boundary operation possible on a point because the boundary of boundary does not exist [26,27]. Consider a 3-simplex represented by four nodes 1, 2, 3, and 4 as shown in Fig. 2. For example, boundary operator operating on the (1-simplex) line 1-2 gives the boundary of that line, which are nodes 1 and 2. It is worth noticing that the boundary operator reduces the dimensionality of the topological objects by one. That is, when operated on a surface or a volume, we get the enclosing lines or surfaces, respectively as results. These are illustrated in Fig. 2 using different colours. The coboundary operator operates on the cochains, which are physical quantities explained in the previous section. The coboundary operator *operates* on the node potentials to give the potential difference between the nodes (electromotance). When it operates on the potential difference on a chain of lines forming a contour, then we get the flux passing through the surface enclosed by the contour. In that sense, the coboundary operator does the opposite of what the boundary operator does - increases the dimensionality of the cochains by one. For more discussion on the ATM framework, please refer to [15,18,28].

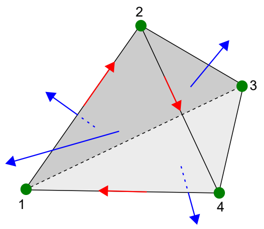

Figure 2. Boundary operators *operating* on 1-simplex (line), 2-simplex (surface), and 3-simplex (volume) gives boundary of those simplexes shown in green, red, and blue colours, respectively.

4 ATM formulation for electromagnetics

The ATM toolset consisting of boundary and coboundary operations acting on chains and cochains, respectively enable us to directly describe the underlying physics of electromagnetics close to experimentation. The 4+1 electromagnetic equations derived using the ATM framework are given below [15,29],

$$\begin{array}{rcll}
\Phi(\partial s^3,\tilde{t}) & = & 0 & (1)\\
\Psi(\partial \tilde{s}^3,t) & = & Q_c(\tilde{s}^3,t) & (2)\\
\mathscr{V}(\partial s^2,\tilde{\tau}) & = & \Phi(s^2,\tilde{t}^-)-\Phi(s^2,\tilde{t}^+) & (3)\\
\mathscr{U}(\partial \tilde{s}^2,\tau) & = & Q_f(\tilde{s}^2,\tau)+\Psi(\tilde{s}^2,t^+)-\Psi(\tilde{s}^2,t^-) & (4)\\
Q_f(\partial \tilde{s}^3,\tau) & = & Q_c(\tilde{s}^3,t^-)-Q_c(\tilde{s}^3,t^+)\,. & (5)
\end{array}$$

The notations used in the above equations are defined as in [15]. Eqn. 1 and Eqn. 2 are the ATM formulations for the Gauss magnetic and electric divergence equations, respectively. Eqn. 3 and Eqn. 4 correspond to Faraday and Ampere laws, respectively. Eqn. 5 corresponds to the electric charge continuity equation. The above ATM formulation for electromagnetics is directly derived from the experimental principles using only physically measurable quantities and completely avoiding differential equations as shown in Fig. 3. We can use the same approach to also derive the ATM formulation for other multiphysics problems like thermodynamics, thermoelectric, quantum tunnelling, etc.

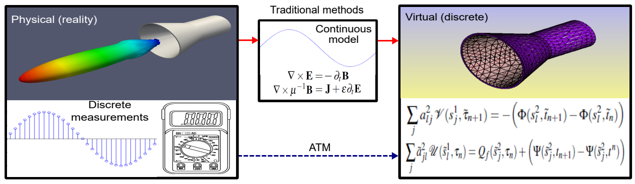

Figure 3. Direct discrete ATM formulations using only physically measurable scalar quantities and without requiring differential equations and field vectors.

5 Further research and applications

Accurate domain truncation techniques such as perfectly matched layers (PML) [30–32] and absorbing boundary conditions (ABC) [33, 34] are topics of further research in the development of ATM. We are currently expanding these boundary truncation techniques for ATM applications, which are available for many conformal time-domain methods [35–43]. Some recent applications of ATM tools in biomedicine [44], thermoelectrics [45], quantum tunnelling [46], radar remote sensing [47] are worth mentioning. The ATM is rather a non-mainstream approach and several efforts are needed to further expand its capabilities. Though the method emerges from a different starting point, there is a strong analogy between differential-calculus-based methods and ATM [48]. The numerical accuracy of ATM is comparable to that of standard FDTD method on structured grids. The full power of ATM lies in its suitability to be used on highly unstructured inhomogeneous grids. Another interesting area of future research is in the comparison of ATM with the recently developed higher-order discontinuous Galerkin method, which will test the limits of this tool for high precision applications [49–51].

6 Summary

The multiscale and multiphysics capabilities of ATM formulation are ideal for modelling various advanced real-world problems. Unlike traditional methods, we showed how the ATM approach can lead to an elegant and direct discrete formulations using only physically measurable scalar quantities for modelling different electromagnetic problems. We have highlighted areas of current and future research in this domain.

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